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Faster Electromagnetic Field Computation Using the Voronoi-Delaunay Transformation

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Abstract– Previously, we have proposed a locally orthogonal discretization method. This method is based on a geometrical duality between the Voronoi polygons and Delaunay triangles so that only one type of potential is required to implement the dual energy approach. In this paper, we propose a Voronoi-Delaunay transformation method to implement the dual energy approach in an ultimate efficient manner.

I. INTRODUCTION

In order to evaluate the electromagnetic fields in a most efficient manner, a dual energy finite element method was proposed [1–3]. The traditional dual energy method requires the use of the two different types of potentials, viz., vector and scalar, and as such the method provides the improved functionals; however, it does not provide the improved local solutions. To overcome this difficulty, we have previously proposed a locally orthogonal discretization method [3,4]. This method is based on a geometrical duality between the Voronoi polygons and Delaunay triangles so that only one type of potential is required to implement the dual energy approach. Thereby, both of the improved functionals as well as local solutions can be obtained by the dual energy approach. Even if a single type of potential is employed to implement the dual energy approach, this locally orthogonal discretization method is compelled to solve the two independent systems, i.e., Voronoi and Delaunay.

In the present paper, we exploit the Voronoi-Delaunay transformation method so that the solution of Delaunay system can be obtained by transforming the solution of Voronoi system. This means that only the Voronoi system of equations has to be solved to implement the dual energy approach. Thus, it is revealed that the electromagnetic fields can be computed in an ultimately efficient manner by the Voronoi-Delaunay transformation method. Some examples demonstrate how highly accurate electromagnetic fields can be computed from a small system by our Voronoi-Delaunay transformation method.

II. THE VORONOI-DELAUNAY TRANSFORMATION METHOD

A. Basic Equations

In two dimensional xy-plane, most of the magnetodynamic fields are reduced to solve the following governing equation

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial x^2} + \frac{1}{\mu}\frac{\partial^2 A}{\partial y^2} - \sigma\frac{\partial A}{\partial t} = -J_s \tag{1}$$

where A, J_s, μ, and σ are the z-components of vector potential, source current density, permeability and conductivity, respectively. The vector potential $\overline{A}$ is related with the flux density $\overline{B}$ by

$$\nabla \times \overline{A} = \overline{B} \tag{2}$$

so that the electric field intensity $\overline{E}$ is given by

$$\overline{E} = -\nabla\phi - \frac{\partial \overline{A}}{\partial t} \tag{3}$$

The scalar potential ϕ and time derivative term $\partial\overline{A}/\partial t$ in (3) are respectively related with the source current density $\overline{J}_s$ and eddy current density $\overline{J}_e$ as

$$\overline{J}_s = -\sigma\nabla\phi \tag{4}$$

$$\overline{J}_e = \sigma(\partial\overline{A}/\partial t) \tag{5}$$

B. Locally Orthogonal Discretization

The key concept of locally orthogonal discretization is to exploit the geometric duality that exists between Delaunay triangles and Voronoi polygons. Delaunay triangles and Voronoi polygons are related by the fact that vertices of Voronoi polygons are the circumcenters of Delaunay triangles. Figure 1 shows the triangles in a Delaunay mesh. The Voronoi polygons associated with these Delaunay triangles are shown by dashed line in Fig. 1. One of the features of this Voronoi-Delaunay diagram is that the sides of the Voronoi polygons are always perpendicular to the sides of the Delaunay triangles [5]. This relationship leads to a locally orthogonal coordinate system as shown Fig. 1. When the vertices i, j of the Delaunay triangle and the vertices k, l of the Voronoi polygon are chosen as node points, following interpolating function may be assumed:

$$A = a_0 + a_1 x + a_2 y + a_3 xy \tag{6}$$

Applying (6) to the node points i, j, k, l in Fig. 1 yields

$$\begin{bmatrix} A_i \\ A_j \\ A_k \\ A_l \end{bmatrix} = \begin{bmatrix} 1 & 0 & a/2 & 0 \\ 1 & 0 & -a/2 & 0 \\ 1 & -b & 0 & 0 \\ 1 & c & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{7}$$

However, it is difficult to represent the coefficients a_0, a_1, a_2, a_3 in terms of the nodal variables A_i, A_j, A_k, A_l, because the determinant of (7) is zero. This means that two complete but independent sets of nodal variables must be defined: one is located at the vertices of the Delaunay triangles and the other is located at the vertices of the Voronoi polygons.

A simple Lagrange interpolation between the nodes i and j in Fig. 1 is given by

$$A_D = (1/2)\left(A_i + A_j\right) + \left(A_i - A_j\right)(y/a) \tag{8}$$

where a is a distance between the nodes i and j. Equation (8) is an interpolating function of the Delaunay system. This interpolating function (8) satisfies the flux density $B_x\,(=\partial A/\partial y)$ continuity between the adjacent Delaunay triangles when each of the triangles takes a distinct permeability μ.

On the other side, a field intensity $H_y\,(= -(1/\mu)\partial A/\partial x)$ must be common between the adjacent Delaunay triangles. This boundary condition can be satisfied by selecting the following interpolating functions for the Voronoi system:

$$A_V = \{(A_k/\mu_1)\,c + (A_l/\mu_2)\,b + (A_l - A_k)\,(x/\mu_2)\} \,/\, \{(c/\mu_1) + (b/\mu_2)\}, \quad -b \le x \le 0 \tag{9a}$$

$$A_V = \{(A_k/\mu_1)\,c + (A_l/\mu_2)\,b + (A_l - A_k)\,(x/\mu_1)\} \,/\, \{(c/\mu_1) + (b/\mu_2)\}, \quad 0 \le x \le c \tag{9b}$$

where the distances b and c are shown in Fig. 1.

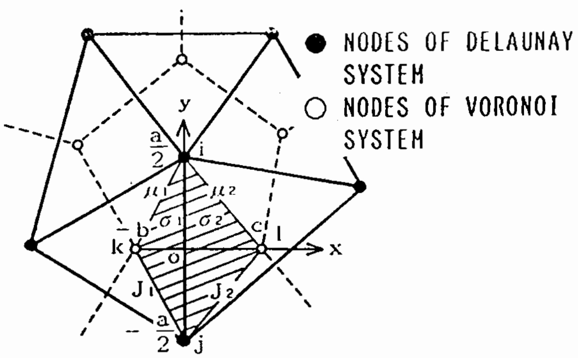

Figure 1. Voronoi-Delaunay diagram and a locally orthogonal coordinate system.

According to these two independent interpolating functions (8) and (9a) or (9b), the governing equation (1) may be reduced to a one-dimensional equation in either the Delaunay or Voronoi sets of variables in the locally orthogonal coordinates:

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial x^2} = \frac{1}{2}\left[\sigma\frac{\partial A}{\partial t} - J_s\right] \tag{10a}$$

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial y^2} = \frac{1}{2}\left[\sigma\frac{\partial A}{\partial t} - J_s\right] \tag{10b}$$

C. Functional and System Equations

A functional satisfying the flux density $B_x (= \partial A/\partial y)$ continuity condition between the adjacent Delaunay triangles is given by

$$F(A) = \iint \left\{ \frac{1}{\mu} \left(\frac{\partial A}{\partial y} \right)^2 + \left(\frac{\sigma}{2} \frac{\partial A}{\partial t} - J_s \right) A \right\} dx dy \tag{11}$$

After substituting (8) into (11), and integrating over the region enclosed by a line i-k-j-l in Fig. 1, we can obtain the functional for the Delaunay system. By taking a minimum of this functional, it is possible to obtain the nodal equations for the Delaunay system. For example, a nodal equation for the node i in Fig. 1 is given by

$$\begin{aligned} &\left(\frac{1}{\mu_1} \frac{b}{a} + \frac{1}{\mu_2} \frac{c}{a} \right) (A_i - A_j) + \frac{a}{48} (b\sigma_1 + c\sigma_2) \frac{\partial}{\partial t} (7A_i + 5A_j) \\ &\quad = \frac{a}{4} (bJ_1 + cJ_2) \end{aligned} \tag{12}$$

Thus, an entire Delaunay system of equations is represented by

$$D_D \Phi_D + E_D (d/dt) \Phi_D = F_D \tag{13}$$

where D_D, E_D are the coefficient matrices corresponding to the first and second terms on the left of (12); F_D is an input current vector corresponding to the right of (12); and Φ_D is the potential vector of Delaunay system, respectively.

On the other side, a functional satisfying the common field $H_y (= -(1/\mu)\ \partial A/\partial x)$ condition between the adjacent Delaunay triangles is given by

$$G(A) = -\iint \left\{ \mu \left(\frac{1}{\mu} \frac{\partial A}{\partial x} \right)^2 + \left(\frac{\sigma}{2} \frac{\partial \hat{A}}{\partial t} - J_s \right) \hat{A} \right\} dx\, dy \tag{14}$$

where $\hat{}$ refers to the prescribed values. After substituting (9a) and (9b) into (14), and integrating over the region enclosed by the line i-k-j-l in Fig. 1, we can obtain the functional for the Voronoi system. By taking a maximum of this functional, it is possible to obtain the nodal equation for the Voronoi system. For example, the nodal equation for the node k in Fig. 1 is given by

$$\frac{A_k - A_l}{\mu_1 (b/a) + \mu_2 (c/a)} + \frac{ab}{2} \sigma_1 \frac{\partial A_k}{\partial t} = \frac{ab}{2} J_1 \tag{15}$$

Thus, an entire Voronoi system of equations is represented by

$$D_V \Phi_V + E_V (d/dt) \Phi_V = F_V \tag{16}$$

where D_V, E_V are the coefficient matrices corresponding to the first and second terms on the left of (15); F_V is an input source current vector corresponding to the right term of (15); and Φ_V is the potential vector of the Voronoi system, respectively.

D. Voronoi-Delaunay Transformation

The common field H_t condition between the adjacent triangles is always satisfied in the Voronoi system. However, it is obvious that the normal flux densities

B_D to the edges of Delaunay triangles are included in the entire solution of the Voronoi system as shown in Figs. 2(a) and 2(b). The condition of normal flux density B_D continuity between the adjacent Delaunay triangles is essentially satisfied by the nodal variables of the Delaunay system. The original governing equation (1) has been already discretized by the Voronoi discretization method so that the discretized problem region is governed only by a Laplace equation and not the original governing equation (1), viz.,

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial x}+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial y}=0 \tag{17}$$

By means of the Delaunay discretization method (12) and (13), the vector potential A_i in Fig. 2(b) can be represented in terms of the vector potentials of the Voronoi system as

$$\begin{aligned}&\left(\frac{1}{\mu_1}\cot\alpha_1+\frac{1}{\mu_2}\cot\beta_2\right)A_{kl}+\left(\frac{1}{\mu_2}\cot\alpha_2+\frac{1}{\mu_3}\cot\beta_3\right)A_{ln}\\&+\left(\frac{1}{\mu_3}\cot\alpha_3+\frac{1}{\mu_4}\cot\beta_4\right)A_{nq}+\left(\frac{1}{\mu_4}\cot\alpha_4+\frac{1}{\mu_5}\cot\beta_5\right)A_{qr}\\&+\left(\frac{1}{\mu_5}\cot\alpha_5+\frac{1}{\mu_1}\cot\beta_1\right)A_{kr}\\&=\left\{\sum_{j=1}^{5}\left(\frac{1}{\mu_j}\right)(\cot\alpha_j+\cot\beta_j)A_i\right\}\end{aligned} \tag{18}$$

where the angles $\alpha_1 \sim \alpha_5$, $\beta_1 \sim \beta_5$ are shown in Fig. 2(b); and A_{kl}, A_{ln}, A_{nq}, A_{qr}, A_{kr} are the potentials at the inter-selected points of the Voronoi polygon and the Delaunay triangle edges in Fig. 2(b). For example, A_{kl} is given by

$$A_{kl}=\frac{(c/\mu_1)A_k+(b/\mu_2)A_l}{(c/\mu_1)+(b/\mu_2)} \tag{19}$$

By means of (18), the potential vector Φ_D of the Delaunay system can be represented in terms of the connection matrix C and of the Voronoi system potential vector Φ_V by

$$\Phi_D = C\Phi_V \tag{20}$$

The Delaunay system of (13) is transformed into the Voronoi system by means of (20) as

$$C^T D_D C\Phi_V + C^T E_D C(d/dt)\Phi_V = C^T F_D \tag{21}$$

where a superscript T refer to the transposed matrices. Thereby, a resultant Voronoi system of equations becomes

$$D\Phi_V + E(d/dt)\Phi_V = F \tag{22}$$

where

$$D=(1/2)\left\{C^T D_D C + D_V\right\} \tag{23a}$$

$$E=(1/2)\left\{C^T E_D C + E_V\right\} \tag{23b}$$

$$F=(1/2)\left\{C^T F_D C + F_V\right\} \tag{23c}$$

The coefficient $1/2$ in (23a)–(23c) is required because a simple summation of (16) and (21) duplicates the input source current vector.

The Delaunay and Voronoi systems respectively satisfy the normal flux density continuity and the common field intensity conditions between the adjacent Delaunay triangles. However, (22) satisfies both of the flux density and field intensity conditions. Therefore, the functional obtained from (22) is a promising one, but the solution vector obtained from (22) is the solution vector of Voronoi system. In order to obtain the improved potentials, a midpoint potential vector Φ_M located at the midpoints between the vertices of the Voronoi polygon and the Delaunay triangle is defined by

$$\Phi_M = C_{DM}\Phi_D + C_{VM}\Phi_V \tag{24}$$

where C_{DM}, C_{VM} are the interpolating matrices between the vertices of Voronoi polygon and the Delaunay triangle. For example, a midpoint potential A_m in Fig. 2(c) is given by

$$A_m = (1/2)A_i + (1/2)A_k \tag{25}$$

by means of (20), (24) can be reduced to

$$\Phi_M = (C_{DM}C + C_{VM})\Phi_V \tag{26}$$

The potential vector Φ_M in (26) is obviously improved in accuracy because the vector Φ_M takes into account the boundary conditions of tangential field intensity and normal flux density in Fig. 2(a).

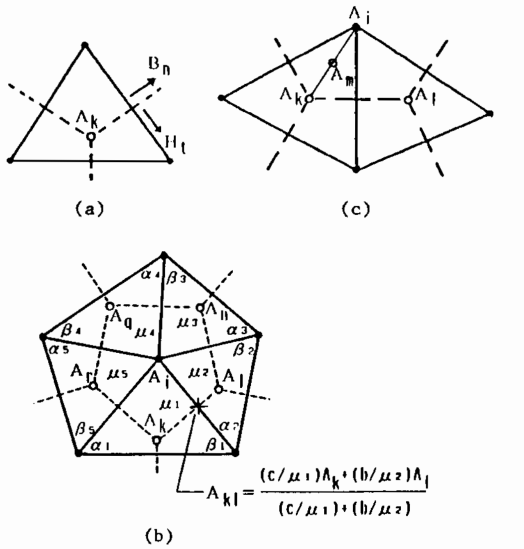

Figure 2. (a) The normal component of flux density B_n and tangential component of field intensity H_t.
(b) Transformation from the Voronoi nodal variables to the Delaunay nodal variables.
(c) Location of the midpoint potential A_m.

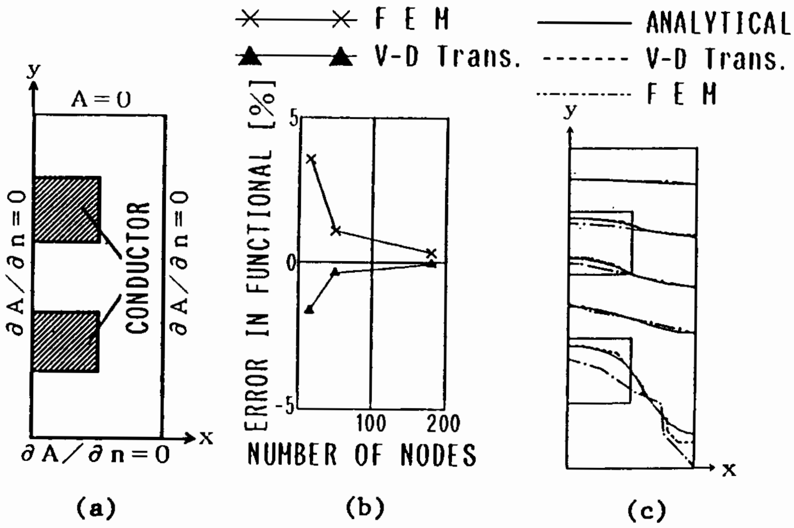

Figure 3. (a) Model of open slot.
(b) Convergence property of the functionals.
FEM: first order triangular finite element method,
V-D Trans. : Voronoi-Delaunay transformation method.
(c) Field distribution in the open slot.
V-D Trans. : 10 Nodal variables,
FEM: 15 Nodal variables.

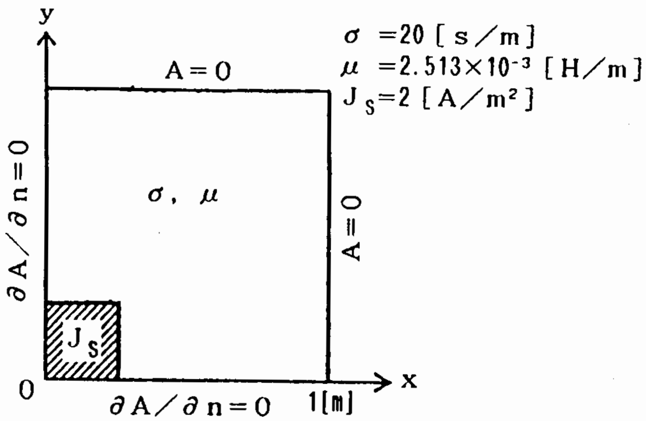

Figure 4. Model of highly permeable conductor.

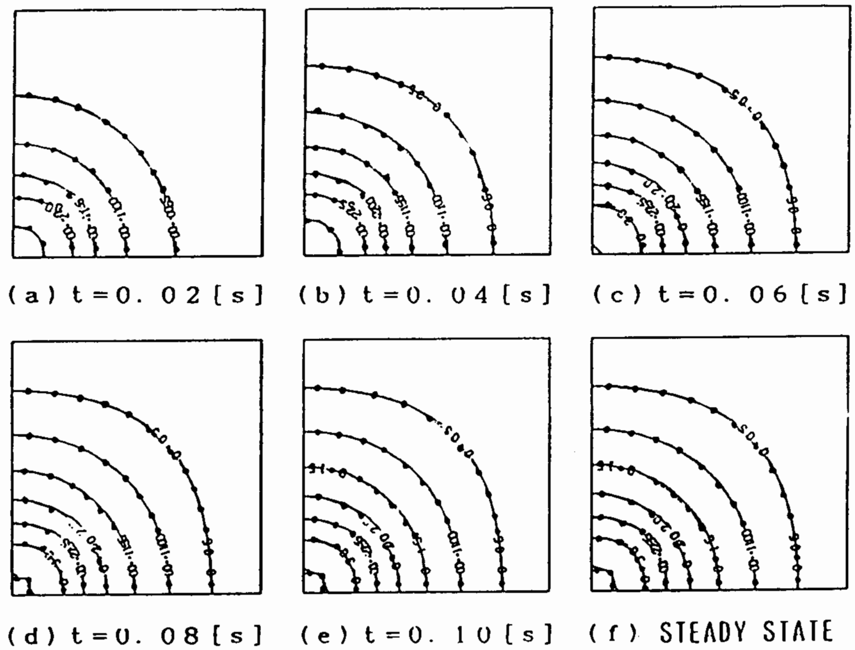

Figure 5. Dynamic field distribution in the highly permeable conductor.
—·—·— Voronoi-Delaunay transformation (16 Nodes)
———— First order finite element method (144 Nodes)
Stepwidth in time $\Delta t = 0.01$ [sec].

E. Examples

At first, we computed a static field distribution in an open slot shown in Fig. 3(a). Figure 3(b) shows a convergence property of the functional. Also, Fig. 3(c) shows a field distribution in the open slot. For comparison, we computed the field distribution by the conventional first order triangular finite element method. By considering the results in Figs. 3(b) and (c), it is revealed that the Voronoi-Delaunay transformation method yields an excellent result even if a small number of nodes is employed.

Secondary, we computed a dynamic field distribution on a highly permeable conductor with square cross-section shown in Fig. 4. A step current density J_s was impressed at a central square portion, and the time discretization was carried out by the trapezoidal method. Figure 5 shows the dynamic field distribution on the conductor. The results in Fig. 5 reveal that only the 16 nodal equations of Voronoi-Delaunay transformation method correspond to the 144 nodal equations of conventional first order finite element method.

III. CONCLUSION

As shown above, we have shown that the Voronoi-Delaunay transformation method provides an excellent results even if a small number of nodes is employed. The previously proposed locally orthogonal discretization method was a quite effective method to calculate the electromagnetic field in an efficient manner, but it was compelled to solve the two systems, i.e., Voronoi and Delaunay. This deficiency has been removed by the Voronoi-Delaunay transformation method. Thus, an ultimate efficient method for the electromagnetic fields has been successfully obtained.

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