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NUMERICAL METHODS FOR POLYPHASE INDUCTION MOTORS

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This paper presents some numerical methods for electromechanical systems of polyphase induction motors, taking into account the space harmonic waves.

Notation

a_k	$= a_1, a_2, \ldots, a_k$, arbitrary constants
$\boldsymbol{D}_m$	= friction matrix
d_m	= coefficient of friction (N m sec/rad)
$\boldsymbol{E}[t]$	$= \{E_s \exp(jv_s t), E_s^* \exp(-jv_s t), 0, 0\}$, voltage vector
E_s	= amplitude of the stator impressed voltage
$f(t, y)$	$= (\mathrm{d}/\mathrm{d}t)y$, function of ordinary differential equations
$\boldsymbol{G}[\theta(t)]$	$= (\partial/\partial\theta(t))\boldsymbol{L}[\theta(t)]$, torque matrix
$\boldsymbol{G}[\alpha]$	$= \boldsymbol{G}[\theta(t) + \alpha h v_m(t+h)]$, modified torque matrix
h	= stepwidth (sec)
$\boldsymbol{I}[t]$	$= \{i_{sp}, i_{sn}, i_{rp}, i_{rn}\}$, current vector
Δi	$= \boldsymbol{I}[t+h] - \boldsymbol{I}[t]$, increase in current vector
$\boldsymbol{J}_m$	= inertia matrix
j_m	= inertia (N m sec^2/rad)
j	$= \sqrt{-1}$, imaginary unit
k	= positive integer
$\boldsymbol{L}[\theta(t)]$	= inductance matrix
$\boldsymbol{L}[\alpha]$	$= \boldsymbol{L}[\theta(t) + \alpha h v_m(t+h)]$, modified inductance matrix
L_s, L_r	stator and rotor self-inductances, respectively
M_1, M_{19}	fundamental and 19th space harmonic wave mutual inductances, respectively
$\boldsymbol{P}_m[t]$	= force vector
p	= number of pole pairs
$\boldsymbol{R}$	$= \lceil R_s, R_s, R_r, R_r \rfloor$, resistance matrix
R_s, R_r	stator and rotor resistances, respectively
$\boldsymbol{S}[\theta(t), v_m(t)]$	$= -\boldsymbol{L}^{-1}[\theta(t)]\boldsymbol{E}[t]$, input vector of the electrical state equation
$\boldsymbol{S}_m$	$= -j_m^{-1} d_m$, coefficient of the mechanical state equation
T	= torque (N m)
$\boldsymbol{U}[\theta(t), t]$	$= \boldsymbol{L}^{-1}[\theta(t)]\boldsymbol{E}[t]$, input vector of the electrical state equation
$\boldsymbol{U}_m[\theta(t), \boldsymbol{I}[t]]$	$= j_m^{-1}(p^2/2)(\boldsymbol{I}^t[t])^* \, \boldsymbol{G}[\theta(t)]\boldsymbol{I}[t]$, input vector of the mechanical state equation
$v_m(t)$	$= (\mathrm{d}/\mathrm{d}t)\theta(t)$, mechanical angular velocity (rad/sec)
vp	= predicted value of the mechanical angular velocity
vc	= corrected value of the mechanical angular velocity
Δv_m	$= v_m(t+h) - v_m(t)$, increase in mechanical angular velocity
$\boldsymbol{Z}[\theta(t), v_m(t)]$	$= \boldsymbol{R} + v_m(t)\boldsymbol{G}[\theta(t)] + \boldsymbol{L}[\theta(t)](\mathrm{d}/\mathrm{d}t)$, impedance matrix
α	= parameter of the linearized method

β = relaxation parameter of the iteration method

$\theta(t)$ = mechanical angle transformed into electrical angle (rad)

Superscript * denotes complex conjugate matrix.

Subscripts a, b, c refer to the stator a-phase, b-phase, c-phase quantities; s, r refer to the stator and rotor; and p, n, m refer to the positive phase, negative phase and mechanical quantities, respectively.

1. Introduction

If an induction motor is to be used in variable speed systems, it becomes necessary to know its dynamic performance.

Numerous studies of the dynamic performance of induction motors have been reported and covered a wide range of performance aspects including starting, overspeeding, switching etc. (e.g. [1]–[5]). The electromechanical systems of equations are nonlinear, and hence difficult, to solve analytically. Therefore, most studies have obtained the solutions either by means of computers or by analytical methods for the linearized equations. Moreover, the electrical systems of equations in these studies have been transformed into simultaneous differential equations with constant coefficients by the tensor transformation method. The tensor transformation method is useful to reduce the electrical systems of equations to a simpler form, and to work out the analytical solutions effectively [6, 7]. However, under somwhat worse conditions such as unbalanced operations and space harmonic waves, the process of tensor transformations becomes too complex and tedious to be practical.

This paper intends to work out an effective and general numerical method which covers various performance aspects, taking into account the space harmonic waves.

An example of an application of the numerical method of this paper is the starting transient performance of a polyphase induction motor. The electrical system of equations taking into account the space harmonic waves are introduced in complex form whose additional details are given in [8,9]. The mechanical system of equations consists of the inertia and the coefficient of friction, both of them include the motor and the load.

For dynamic performance computations of polyphase induction motors, three different kind of numerical methods are examined. One of the methods employs the analytically linearized mathematical model. The second one iterates sequentially between electrical and mechanical systems of equations. The third one is the well known one-step method (auch as Runge-Kutta type methods) based on Taylor series expansion.

2. Mathematical model

Fundamentally, the electromechanical systems of equations involves two systems – one of them is the electrical system and the other one is the mechanical system.

The electrical system of equations is preferably expressed in matrix notation involving a voltage vector $\boldsymbol{E}[t]$ and a current vector $\boldsymbol{I}[t]$. With $\boldsymbol{Z}[\theta(t), v_m(t)]$ denoting the impedance matrix, the electrical system of equations is given as

$$E[t] = Z[\theta(t), v_m(t)]\, I[t] \;, \tag{1}$$

where t, $v_m(t)$ and $\theta(t)$ denote time, mechanical angular velocity and the angle between the stator and the rotor, respectively.

The impedance matrix $Z[\theta(t), v_m(t)]$ is a linear combination of the resistance matrix R, the inductance matrix $L[\theta(t)]$ and the torque matrix $G[\theta(t)]$ (which can be obtained by differentiating the matrix $L[\theta(t)]$ with respect to the angle $\theta(t)$).

This yields

$$Z[\theta(t), v_m(t)] = R + v_m(t)\, G[\theta(t)] + L[\theta(t)]\,(\mathrm{d}/\mathrm{d}t)\,. \tag{2}$$

The output torque T of the electrical system is expressed in terms of the current vector $I[t]$, the torque matrix $G[\theta(t)]$ and the number of pole pairs p:

$$T = (p/2)(I^t[t])^*\, G[\theta(t)]\, I[t] \;, \tag{3}$$

where $(I^t[t])^*$ has elements which are the complex conjugates of those of $I^t[t]$, and $I^t[t]$ is the transpose of $I[t]$.

In general, the matrices R, $G[\theta(t)]$ and $L[\theta(t)]$ in eqs. (2), (3) are square matrices.

The mechanical system of equations is given in terms of the force vector $P_m[t]$, the velocity vector $V_m[t]$, the friction matrix D_m and the inertia matrix J_m, that is

$$P_m[t] = [D_m + J_m(\mathrm{d}/\mathrm{d}t)]\; V_m[t] \;, \tag{4}$$

where a subscript m denotes the mechanical quantities. In the general case, the matrices $P_m[t]$, $V_m[t]$ in eq. (4) are column matrices; and D_m, J_m are square matrices.

The electrical and mechanical systems are coupled by the velocity vector $v_m(t)$ which appears in the velocity matrix $V_m[t]$ and which is a function of the electrical impedance matrix $Z[\theta(t), v_m(t)]$.

For simplicity, it is preferable to consider a concrete example. Therefore, this paper treats an example from the various electrical and mechanical systems.

Each element of the voltage vector $E[t]$ is a state voltage with amplitude E_s and angular velocity v_s, viz.

$$E[t] = \{E_s \exp(jv_s t), E_s^* \exp(-jv_s t), 0, 0\} \;, \tag{5}$$

where j is the imaginary unit ($j = \sqrt{-1}$).

The inductance matrix $L[\theta(t)]$ is classified into four cases by the relations of the number of rotor phases and of pole pairs p as described in [8]. The numerical methods of this paper are applicable to all four cases. In order to illustrate the essential characteristic of the methods, a special case is treated as an example, where only the 19th space harmonic is chosen, because it has a big contributions to abnormal torque as described in [8, 9]. The elements of $L[\theta(t)]$ are the stator self-inductance L_s, the rotor self-inductance L_r, the mutual inductance of fundamental wave M_1 and the mutual inductance of the 19th space harmonic wave M_{19}. The inductance matrix $L[\theta(t)]$ (square matrix of order 4) is

$$L[\theta(t)] = \begin{bmatrix} L_s & 0 & M_1 \exp(j\theta(t)) & M_{19} \exp(j19\theta(t)) \\ 0 & L_s & M_{19} \exp(-j19\theta(t)) & M_1 \exp(-j\theta(t)) \\ M_1 \exp(-j\theta(t)) & M_{19} \exp(j19\theta(t)) & L_r & 0 \\ M_{19} \exp(-j19\theta(t)) & M_1 \exp(j\theta(t)) & 0 & L_r \end{bmatrix} \quad (6)$$

The torque matrix $G[\theta(t)]$ (square matrix of order 4) is given by

$$G[\theta(t)] = (\partial/\partial\theta(t))L[\theta(t)] \ . \quad (7)$$

The resistance matrix R (diagonal matrix of order 4) depends on the stator resistance R_s and the rotor resistance R_r as follows:

$$R = \lceil R_s, R_s, R_r, R_r \rfloor \ . \quad (8)$$

The current vector $I[t]$ (column matrix of order 4) consists of the stator positive phase current i_{sp}, the stator negative phase current i_{sn}, the rotor positive phase current i_{rp} and the rotor negative phase current i_{rn}, viz.

$$I[t] = \{i_{sp}, i_{sn}, i_{rp}, i_{rn}\} \ . \quad (9)$$

The element of the force vector $P_m[t]$ is only the output torque T of the electrical system:

$$P_m[t] = (p/2)(I^t[t])^* \, G[\theta(t)] \, I[t] \ . \quad (10)$$

The friction and inertia matrices involve respectively the inertia j_m and the coefficient of friction d_m; both of them include the motor and the load. They are

$$J_m = j_m \ , \quad (11)$$

$$D_m = d_m \ . \quad (12)$$

The element of the velocity vector $V_m[t]$ is only the mechanical angular velocity $v_m(t)$, and it is given in terms of the number of pole pairs p, because the mechanical angular velocity is transformed into the electrical angular velocity, that is

$$V_m[t] = v_m(t)/p \ . \quad (13)$$

The relation between the mechanical angular velocity $v_m(t)$ and the angle $\theta(t)$ is expressed by

$$(\mathrm{d}/\mathrm{d}t)\theta(t) = v_m(t) \ . \quad (14)$$

For mathematical convenience the complex stator currents i_a, i_b, i_c are defined by the following equation, whose real parts give the original coordinate stator currents (in three-phase stator circuits):

$$\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-j2\pi/3) & \exp(-j4\pi/3) \\ 1 & \exp(-j4\pi/3) & \exp(-j2\pi/3) \end{bmatrix} \begin{bmatrix} 0 \\ i_{sp} \\ i_{sn} \end{bmatrix} , \tag{15}$$

where the stator phase are represented by suffixes a, b and c.

3. Numerical methods of solution

3.1. Linearized method. By means of eqs. (1)–(14) it is possible to write the electromechanical system of equations as

$$\boldsymbol{L}[\theta(t)]\,(\mathrm{d}/\mathrm{d}t)\boldsymbol{I}[t] = -[\boldsymbol{R} + v_m(t)\,\boldsymbol{G}[\theta(t)]\,]\,\boldsymbol{I}[t] + \boldsymbol{E}[t] , \tag{16}$$

$$(j_m/p)(\mathrm{d}/\mathrm{d}t)v_m(t) = -(d_m/p)v_m(t) + (p/2)(\boldsymbol{I}^t[t])^*\,\boldsymbol{G}[\theta(t)]\,\boldsymbol{I}[t] . \tag{17}$$

This set of equations is a system of nonlinear simultaneous differential equations which is linearized by finite difference methods, namely

$$\boldsymbol{L}[\alpha]\,[\boldsymbol{I}[t+h] - \boldsymbol{I}[t]\,]/h = -[\boldsymbol{R} + \{(1-\alpha)v_m(t) + \alpha v_m(t+h)\}\,\boldsymbol{G}[\alpha]\,]\,[(1-\alpha)\boldsymbol{I}[t] + \alpha\boldsymbol{I}[t+h]\,] + \boldsymbol{E}[t+\alpha h] , \tag{18}$$

$$j_m\,[v_m(t+h) - v_m(t)]/h = -d_m\,\{(1-\alpha)\,v_m(t) + \alpha v_m(t+h)\} + (p^2/2)\{[(1-\alpha)\boldsymbol{I}[t] + \alpha\boldsymbol{I}[t+h]\,]^t\}^*\,\boldsymbol{G}[\alpha]\,[(1-\alpha)\boldsymbol{I}[t] + \alpha\boldsymbol{I}[t+h]\,] , \tag{19}$$

where the matrices $\boldsymbol{L}[\alpha]$ and $\boldsymbol{G}[\alpha]$ in eqs. (18), (19) are defined by

$$\boldsymbol{L}[\alpha] = \boldsymbol{L}[\theta(t) + \alpha h v_m(t+h)] , \tag{20}$$

$$\boldsymbol{G}[\alpha] = \boldsymbol{G}[\theta(t) + \alpha h v_m(t+h)] . \tag{21}$$

The parameter α in eqs. (18)–(21) can be chosen arbitrarily e.g. $\alpha = 0$, $\alpha = 1/2$ and $\alpha = 1$ yield forward, central and backward differences, respectively. It is assumed that the increase in the current vector $\boldsymbol{I}[t+h]$ and mechanical angular velocity $v_m(t+h)$ are denoted Δi and Δv_m and that their products can be neglected, that is

$$\boldsymbol{I}[t+h] = \boldsymbol{I}[t] + \Delta i , \tag{22}$$

$$v_m(t+h) = v_m(t) + \Delta v_m \; , \tag{23}$$

$$(\Delta v_m)(\Delta i) = (\Delta i)^2 = (\Delta v_m)^2 = 0 \; . \tag{24}$$

Then, by considering eqs. (18)–(24), the vectors $\boldsymbol{I}[t+h]$ and $v_m(t+h)$ are obtained as

$$\begin{bmatrix} \boldsymbol{I}[t+h] \\ v_m(t+h) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}[t] \\ v_m(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{L}[\alpha] + \alpha h[\boldsymbol{R} + v_m(t)\boldsymbol{G}[\alpha]\,\boldsymbol{I}[t]] & \alpha h \boldsymbol{G}[\alpha]\,\boldsymbol{I}[t] \\ -\alpha h p^2 (\boldsymbol{I}^t[t])^* \boldsymbol{G}[\alpha] & j_m + \alpha h d_m \end{bmatrix}^{-1}$$

$$\times\, h \begin{bmatrix} -[\boldsymbol{R} + v_m(t)\boldsymbol{G}[\alpha]]\,\boldsymbol{I}[t] + \boldsymbol{E}[t+\alpha h] \\ -d_m v_m(t) + (p^2/2)(\boldsymbol{I}^t[t])^* \boldsymbol{G}[\alpha]\,\boldsymbol{I}[t] \end{bmatrix} . \tag{25}$$

However, it is impossible to obtain the solutions $(\boldsymbol{I}[t+h], v_m(t+h))$ directly from eq. (25) by the application of central ($\alpha = 1/2$) or backward ($\alpha = 1$) difference methods because the mechanical angular velocity $v_m(t+h)$ is a function of $\boldsymbol{L}[\alpha]$ and $\boldsymbol{G}[\alpha]$. Therefore, in a first step, a predicted value of $v_m(t+h)$ is obtained by the forward difference method ($\alpha = 0$) of eq. (25). Afterwards, corrected values of $v_m(t+h)$ and the current vector $\boldsymbol{I}[t+h]$ are computed by the central difference method ($\alpha = 1/2$). The angle $\theta(t+h)$ is given by

$$\theta(t+h) = \theta(t) + h v_m(t+h) \; . \tag{26}$$

3.2. Iteration method. By considering eqs. (5)–(13) with the electromechanical system (1), (4), it is possible to derive the following equations

$$(\mathrm{d}/\mathrm{d}t)\boldsymbol{I}[t] = \boldsymbol{S}[\theta(t), v_m(t)]\,\boldsymbol{I}[t] + \boldsymbol{U}[\theta(t), t] \; , \tag{27}$$

$$(\mathrm{d}/\mathrm{d}t)v_m(t) = \boldsymbol{S}_m v_m(t) + \boldsymbol{U}_m[\theta(t), \boldsymbol{I}[t]] \; . \tag{28}$$

Each matrix in eqs. (27), (28) is defined by

$$\boldsymbol{S}[\theta(t), v_m(t)] = -\boldsymbol{L}^{-1}[\theta(t)]\,[\boldsymbol{R} + v_m(t)\,\boldsymbol{G}[\theta(t)]] \; , \tag{29}$$

$$\boldsymbol{U}[\theta(t), t] = \boldsymbol{L}^{-1}[\theta(t)]\,\boldsymbol{E}[t] \; , \tag{30}$$

$$\boldsymbol{S}_m = -j_m^{-1} d_m \; , \tag{31}$$

$$\boldsymbol{U}_m[\theta(t), \boldsymbol{I}[t]] = j_m^{-1}(p^2/2)(\boldsymbol{I}^t[t])^* \boldsymbol{G}[\theta(t)]\,\boldsymbol{I}[t] \; . \tag{32}$$

If the angle $\theta(t)$ and the mechanical angular velocity $v_m(t)$ in eq. (27) are assumed to be constant, then eq. (27) can be numerically solved as described in [9]. Therefore, in a first step, it is assumed that the predicted value of the angular velocity vp takes the same value as $v_m(t)$. Then a first approximate current vector $\boldsymbol{I}[t+h]$ is calculated by

$$I[t+h] = [I_4 - (h/2)S[\theta(t) + (h/2)vp, vp]]^{-1} \{[I_4 + (h/2)S[\theta(t) + (h/2)vp, vp]] I[t] + hU[\theta(t) + (h/2)vp, t + (h/2)]\} , \quad (33)$$

where I_4 denotes the unit matrix of order 4.

The result ($I[t+h]$) of eq. (33) is substituted into eq. (28), which is the numerically solved by (the same method as described in [10])

$$vc = [1 - (h/2)S_m]^{-1} \{[1 + (h/2)S_m] v_m(t) + hU_m[\theta(t) + hvp, I[t+h]]\} . \quad (34)$$

Then the result (i.e. corrected angular velocity vc) is substituted into eq. (33) with some modifications, namely by using a relaxation parameter β,

$$new(vp) = old(vp) + \beta[vc - old(vp)] . \quad (35)$$

Eq. (35) is substituted into eq. (33). The process is iterated to reduce the discrepancy $|vp - vc|$. The results are the current vector $I[t+h]$ and the mechanical angular velocity $v_m(t+h)$ (computed as the vp or vc). Also the angle $\theta(t)$ can be calculated from eq. (26). Fig. 1 shows the flow chart of this iteration method.

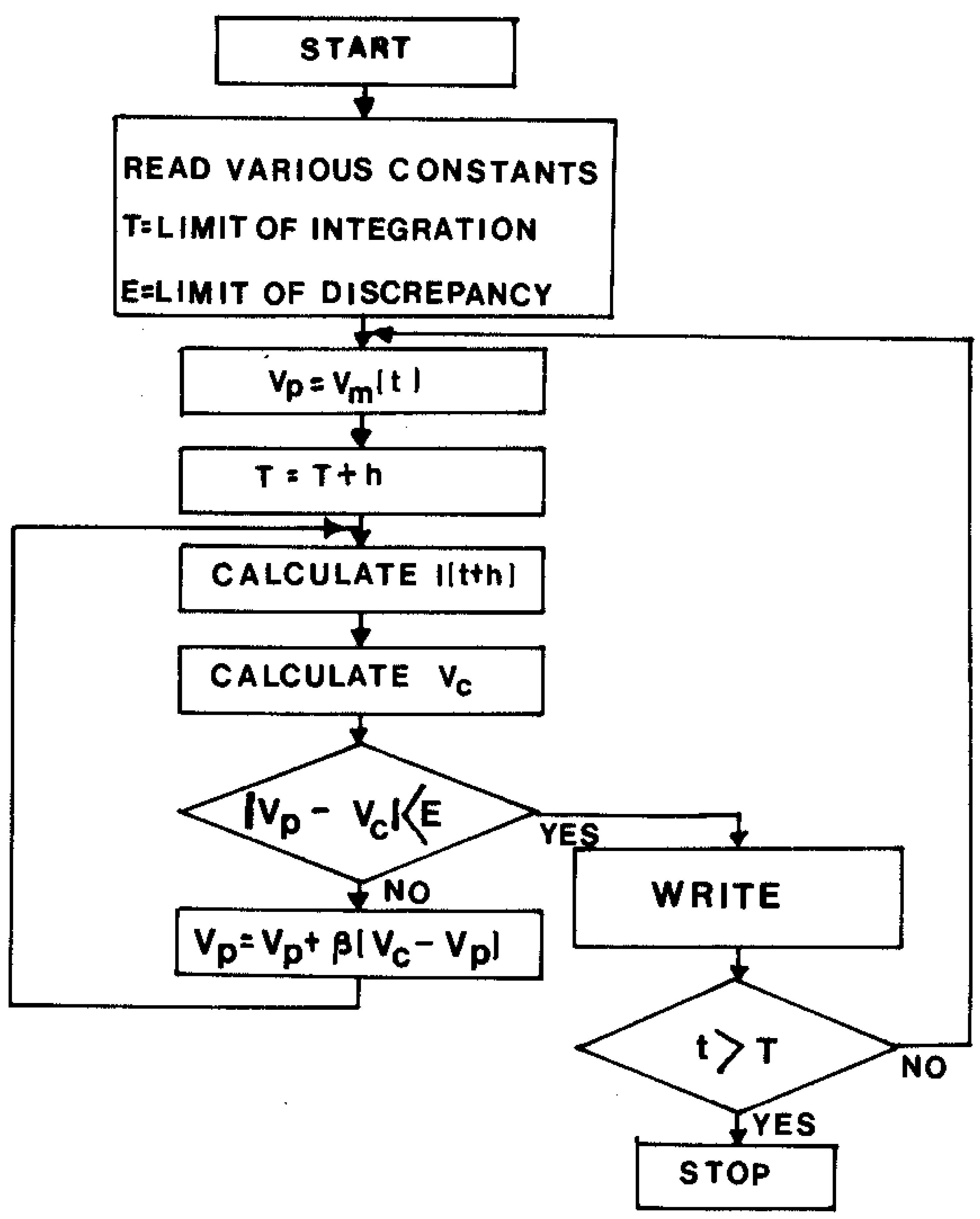

Fig. 1. Flow chart of the iteration method.

3.3. One-step method. It is possible to regard the system of equations of this paper as the following simple equation:

$$(\mathrm{d}/\mathrm{d}t)y = f(t, y)\,. \tag{36}$$

Most numerical methods for the solution of this equation may be put into two classes: predictor-corrector methods, such as Adams and Moulton; and one-step methods based on Taylor series expansion, such as Runge-Kutta type methods [11, 12]. Predictor-corrector methods furnish attractive algorithms because of the relatively small number of derivative evaluations required. However, in order to use predictor-corrector methods, an appropriate number of starting points must be provided in addition to the initial point.

With modern computer the number of derivative evaluations is not such an important matter in the selection of the numerical methods. Because of the simplicity of programming, one-step methods are not only useful for starting point calculations of predictor-corrector methods, but also useful for obtaining the entire solution.

The one-step method of this paper requires a relatively large number of derivative evaluations. However, it is possible to obtain more accurate numerical solutions because the one-step method of this paper calculates each term of the Taylor series expansion of the rigorous solution (see appendix). The general formula is given in the appendix, and the various formulas are given in table 1; the extension of the latter formulas to systems of equations is immediate and obvious. The system of equations treated in this paper are eqs. (14), (27) and (28). Namely, the electromechanical system of equations is rewritten by

Table 1
Formulas for the ordinary differential equation $(\mathrm{d}/\mathrm{d}t)y = f(t, y)$

(1) Third degree	$f_1 = -f_{11} + 4f_{21}$ $f_2 = f_{12}$ $y(t_0 + h) = y_0 + hf_0 + (hf_1/2) + (hf_2/3)$
(2) Fourth degree	$f_1 = 0.5f_{11} - 8f_{21} + 13.5f_{31}$ $f_2 = -f_{12} + 8f_{22}$ $f_3 = f_{13}$ $y(t_0 + h) = y_0 + hf_0 + (hf_1/2) + (hf_2/3) + (hf_3/4)$
(3) Fifth degree	$f_1 = -(f_{11}/6) + 8f_{21} - 40.5f_{31} + (128/3)f_{41}$ $f_2 = 0.5f_{12} - 16f_{22} + 40.5f_{32}$ $f_3 = -f_{13} + 16f_{23}$ $f_4 = f_{14}$ $y(t_0 + h) = y_0 + hf_0 + (hf_1/2) + (hf_2/3) + (hf_3/4) + (hf_4/5)$
(4) Sixth degree	$f_1 = (f_{11}/24) - (16/3)f_{21} + (243/4)f_{31} - (512/3)f_{41} + (3125/24)f_{51}$ $f_2 = -(f_{12}/6) + 16f_{22} - 121.5f_{32} + (512/3)f_{42}$ $f_3 = 0.5f_{13} - 32f_{23} + 121.5f_{33}$ $f_4 = -f_{14} + 32f_{24}$ $f_5 = f_{15}$ $y(t_0 + h) = y_0 + hf_0 + (hf_1/2) + (hf_2/3) + (hf_3/4) + (hf_4/5) + (hf_5/6)$

$f_0 = f(t_0, y_0)$,

$$f_{kj} = f\left(t_0 + h/k, y_0 + \sum_{q=1}^{j} f_{q-1}/[k^q q]\right) - \sum_{q=1}^{j} f_{q-1}/k^{q-1}\,.$$

$$(\mathrm{d}/\mathrm{d}t)\begin{bmatrix} I[t] \\ v_m(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} S[\theta(t), v_m(t)]\, I[t] + U[\theta(t), t] \\ S_m v_m(t) + U_m[\theta(t), I[t]] \\ v_m(t) \end{bmatrix}, \tag{37}$$

where the matrices $S[\theta(t), v_m(t)]$, $U[\theta(t), t]$, S_m, $U_m[\theta(t), I[t]]$ are respectively defined by eqs. (29)–(32).

4. Numerical solution

Among the three numerical methods of this paper, the linearized and iteration methods are based on a central difference method. The numerical solution obtained by a central difference method coincides with the Taylor series expansion of the rigorous solution up to the first three terms (i.e. second degree (h^2) accuracy) [9]. Table 1 shows the various (up to the sixth degree (h^6)) formulas of the one-step method. Then the one-step method is theoretically possible to provide the most accurate numerical solution in the three numerical methods of this paper.

For obtaining the most accurate solution the fifth degree formula in table 1 is selected by the several numerical tests taking a small stepwidth. The relaxation parameter β of the iteration method is selected as $\beta = 1/2$ by the numerical tests when the convergence of the discrepancy $|vp - vc|$ is taken into account.

Table 2
Various constants of the calculated motor
(initial values are all zero)

Voltage	$V_s = \sqrt{2/3}\,200$	(V)
Angular velocity	$v_s = 100\pi$	(rad/sec)
Resistance	$R_s = R_s = 5$	(Ω)
	$L_s = L_r = 0.31831$	(H)
Inductance	$M_1 = 0.30239$	(H)
	$M_{19} = 0.30239/(19 \times 19)$	(H)
Number of pole pairs	$p = 2$	
Coefficient of friction	$d_m = 0.005$	(N m sec/rad)
Inertia	$j_m = 0.02$	(N m sec^2/rad)

Table 2 shows the various constants of the numerical example.

The electromechanical system of equation is numerically solved by (a) the linearized method, (b) the iteration method with a 0.01 percent discrepancy $|vp - vc|$ and (c) the one-step method.

Fig. 2 shows that the results of these numerical methods are in fairly good agreement. It is found that the iteration method required an average of seven iterations. The fifth degree (h^5) terms of the stator positive phase currents i_{sp} obtained by the one step method (see appendix) were always smaller than 10^{-8}.

To summarize this section, any numerical methods examined here are applicable to the electro-

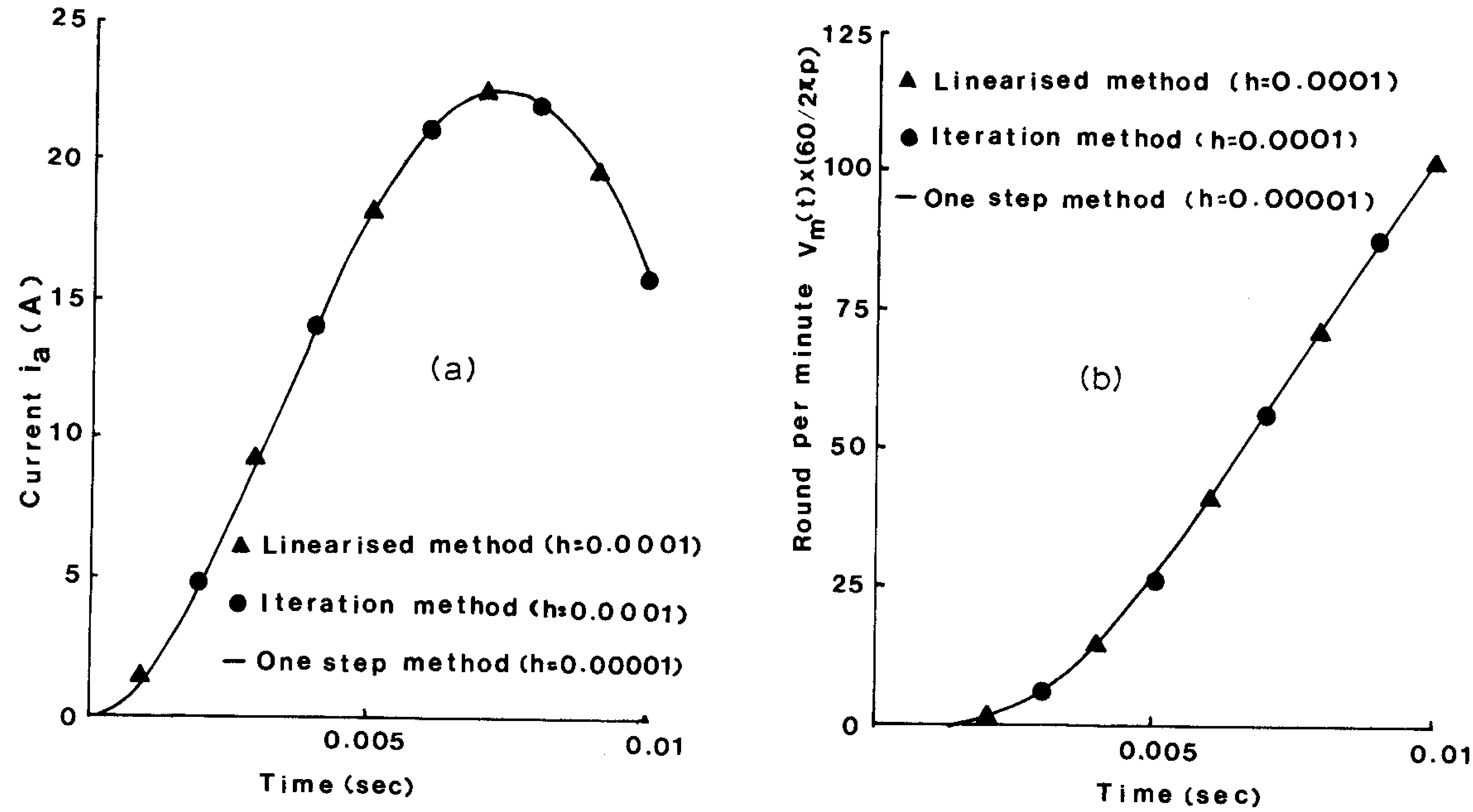

Fig. 2. Comparison of the results of various numerical methods: (a) Stator a-phase current, (b) Mechanical angular velocity in term of round per minute.

nechanical systems of equations for any performance aspects. The one step-method is possible to provide the most accurate numerical solution compared with those of other numerical methods. But it requires a large amount of operation counts because of the large number of derivative evaluations (i.e. eleventh derivative evalations). The iteration method displays its ability within small

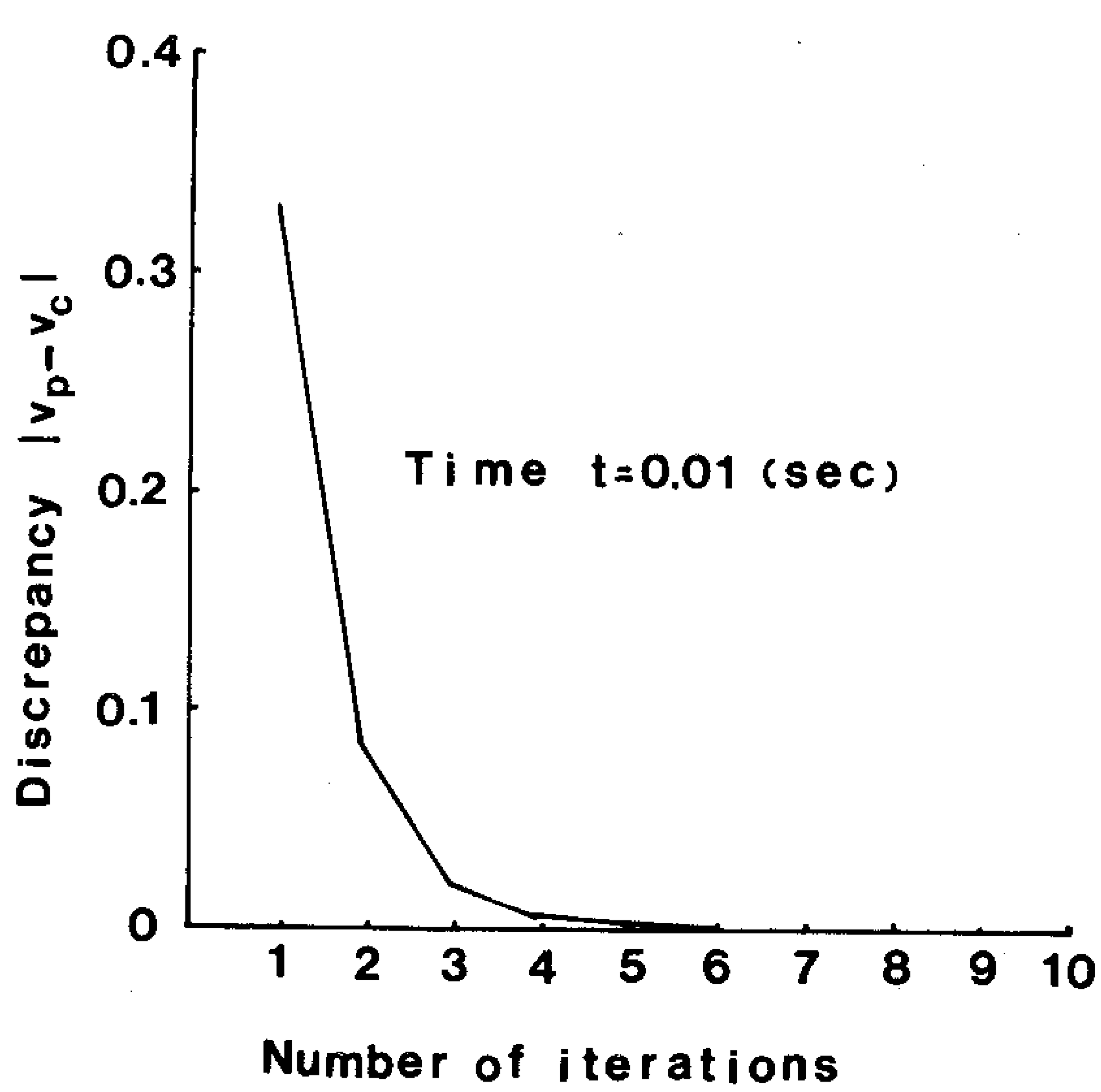

Fig. 3. Convergence process of the iteration method.

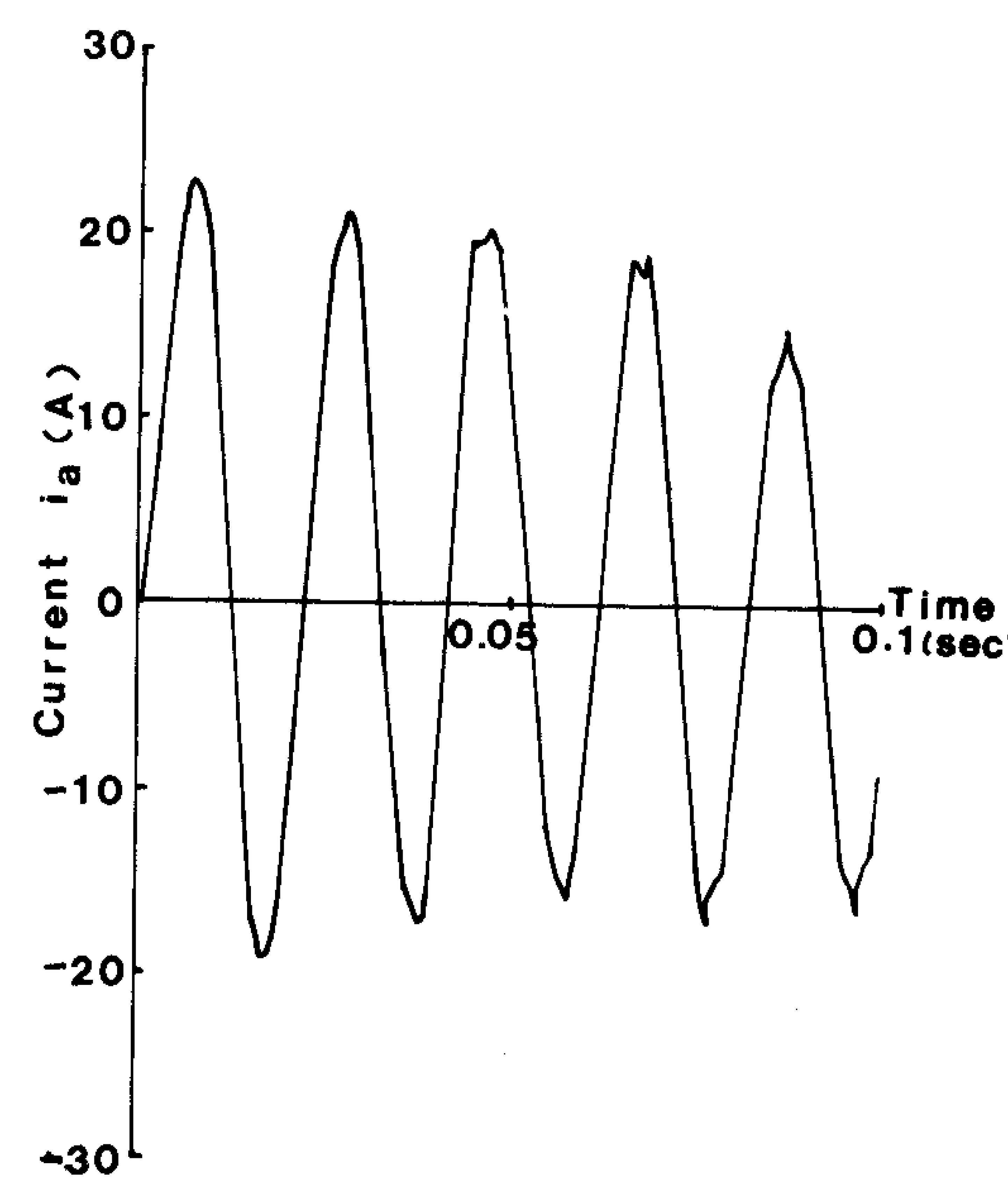

Fig. 4a.

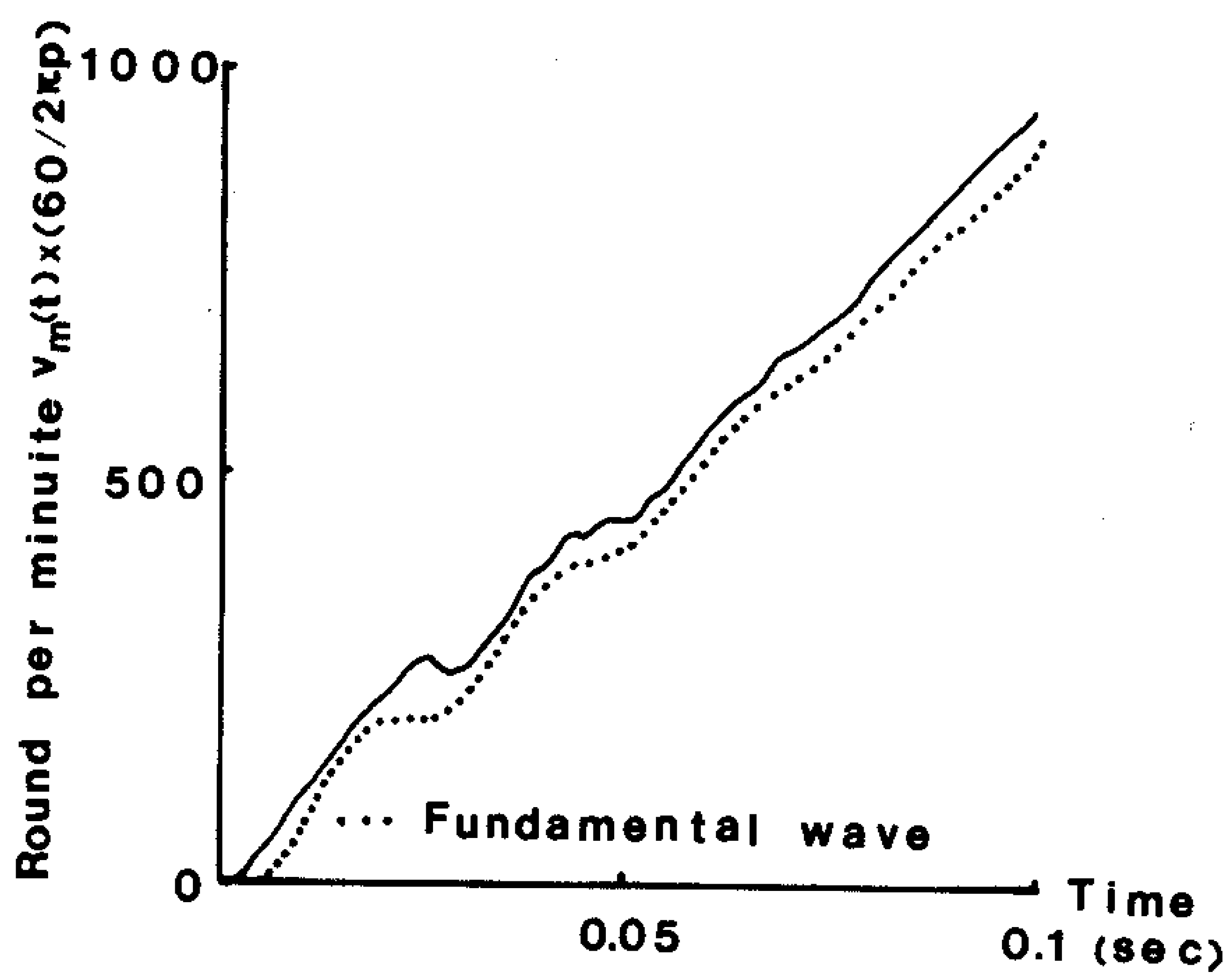

Fig. 4b.

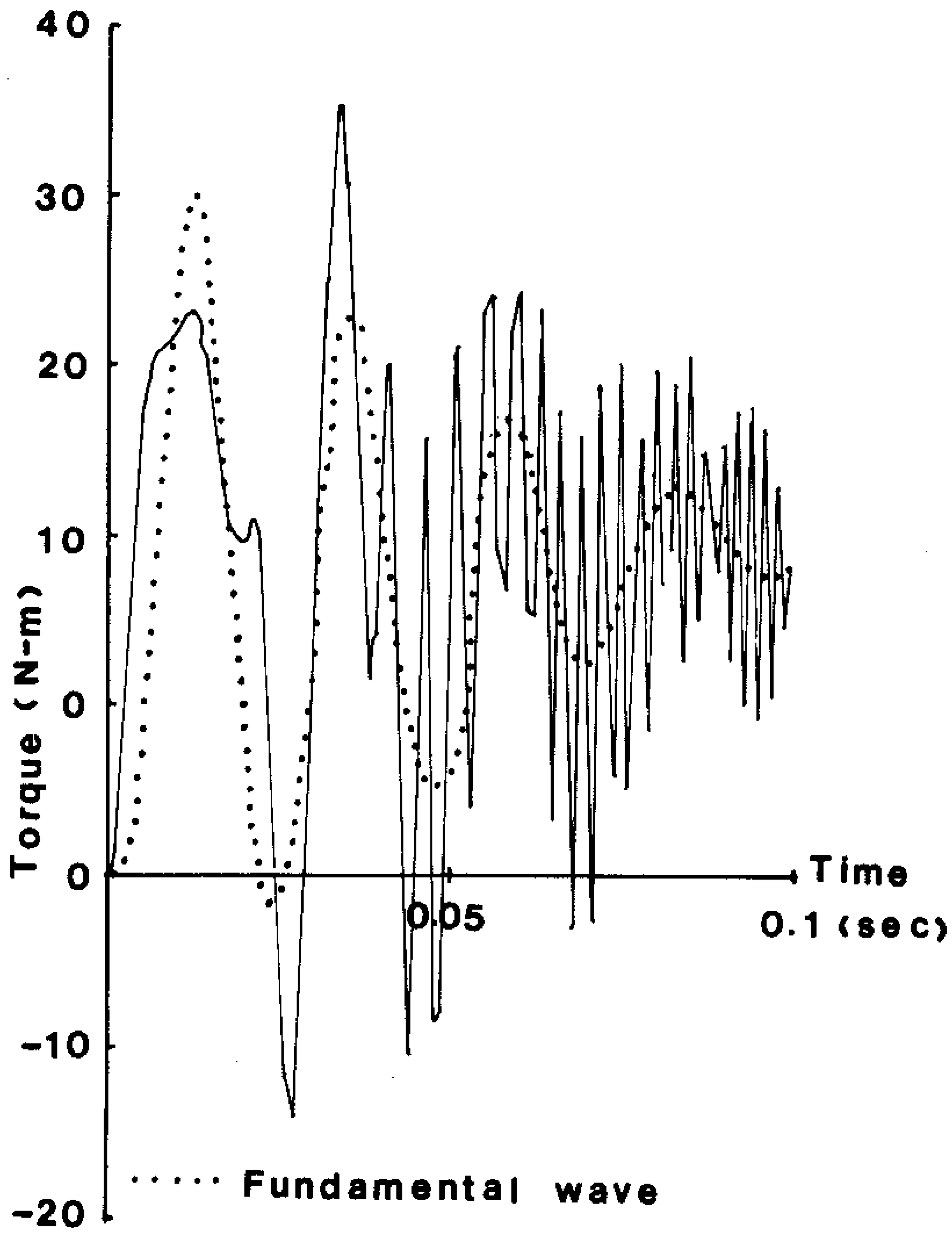

Fig. 4c.

Fig. 4. Examples of numerical solutions computed by the linearized method using the stepwidth $h = 0.0001$: (a) Stator a-phase current, (b) Mechanical angular velocity in term of round per minute, (c) Starting transient torque.

varying solutions as regards the time variation. Fig. 3 shows an example of the convergence process of the iteration method. The linearized method requires some manual effort in the programming, but it is the most reliable and speedy method. Some examples of the numerical solutions computed by the linearized method are shown in fig. 4.

5. Conclusion

This paper presents three different numerical methods for the electromechanical system of polyphase induction motors. The linearized method requires a considerable manual effort to write a computer program. However, such a program has a considerable generality for the computations of various performance aspects and provides reliable solutions with a low operation count. Therefore, the linearized method is the most promising one.

Within small varying numerical solutions as regards the time variation, the operation count of the iteration method is much the same as that of the linearized method. The operation count of the one-step method (fifth degree formula in table 1) requires about 5 times that of the linearized method.

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The practical computations given in this paper were carried out on the computer **FACOM** 230-45S of Hosei University.

Appendix. One-step method

The solution $y(t_0 + h)$ of the differential equation

$$(\mathrm{d}/\mathrm{d}t)y = f(t, y) \tag{A.1}$$

can be given by means of a Taylor series expansion. Neglecting the fourth degree (h^4) terms, the solution of eq. (A.1) is

$$y(t_0 + h) = y_0 + hf_0 + (h^2/2)f' + (h^3/6)f'' , \tag{A.2}$$

where t_0 and y_0 are respectively the initial point and the initial value, and the other terms in eq. (A.2) are

$$f_0 = f(t_0, y_0) , \tag{A.3}$$

$$f = f(t, y) , \tag{A.4}$$

$$f' = \{(\partial/\partial t) + f(\partial/\partial y)\} f_{\substack{t=t_0\\ y=y_0}} , \tag{A.5}$$

$$f'' = \{(\partial/\partial t) + f(\partial/\partial y)\}^2 f_{\substack{t=t_0\\ y=y_0}} + \{\partial f/\partial y\}_{\substack{t=t_0\\ y=y_0}} f' . \tag{A.6}$$

Then, with a_1 and a_2 denoting arbitrary (but not equal) constants, the third and fourth terms of the Taylor series expansion (A.2) can be obtained from

$$(h^2/2)f' = \frac{h}{2}\,\frac{\begin{vmatrix} f(t_0 + a_1 h, y_0 + a_1 hf_0) - f_0 & a_1^2 \\ f(t_0 + a_2 h, y_0 + a_2 hf_0) - f_0 & a_2^2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_1^2 \\ a_2 & a_2^2 \end{vmatrix}} , \tag{A.7}$$

$$(h^3/6)f'' = (h/3)(1/a_1^2)\{f(t_0 + h, y_0 + a_1 hf_0 + [(a_1 h)^2/2] f') - f_0 - a_1 hf'\} . \tag{A.8}$$

Eqs. (A.7), (A.8) are based on the following relations:

$$f(t_0 + a_1 h, y_0 + a_1 h f_0) = f_0 + a_1 h f' + [(a_1 h)^2/2]\ \{(\partial/\partial t) + f_0(\partial/\partial y)\}^2 f_{\substack{t=t_0\\ y=y_0}} + \dots , \tag{A.9}$$

$$f(t_0 + a_2 h, y_0 + a_2 h f_0) = f_0 + a_2 h f' + [(a_2 h)^2/2]\ \{(\partial/\partial t) + f_0(\partial/\partial y)\}^2 f_{\substack{t=t_0\\ y=y_0}} + \dots , \tag{A.10}$$

$$f(t_0 + a_1 h, y_0 + a_1 h f_0 + [(a_1 h)^2/2] f') = f_0 + a_1 h f' + [(a_1 h)^2/2] f'' + \dots . \tag{A.11}$$

Therefore, each term of the Taylor series expansion (A.2) is numerically obtainable, so that eq. (A.1) is solved with third degree (h^3) accuracy.

The process described above is extended to the process of $(i+1)$th degree solution, that is

$$\left.\begin{aligned}
&f_0 = f(t_0, y_0) ,\\
&f_{kj} = f\Big(t_0 + a_k h, y_0 + h \sum_{q=1}^{j} a_k^q f_{q-1}/q\Big) - \sum_{q=1}^{j} a_k^{q-1} f_{q-1} ,\\
&f_j = \frac{\begin{vmatrix} f_{1j} & a_1^{j+1} & \cdot & a_1^{i-j+1} \\ f_{2j} & a_2^{j+1} & \cdot & a_2^{i-j+1} \\ \cdot & \cdot & \cdot & \cdot \\ f_{i-j+1,j} & a_{i-j+1}^{j+1} & \cdot & a_{i-j+1}^{i-j+1} \end{vmatrix}}{\begin{vmatrix} a_1^{j} & a_1^{j+1} & \cdot & a_1^{i-j+1} \\ a_2^{j} & a_2^{j+1} & \cdot & a_2^{i-j+1} \\ \cdot & \cdot & & \cdot \\ a_{i-j+1}^{j} & a_{i-j+1}^{j+1} & \cdot & a_{i-j+1}^{i-j+1} \end{vmatrix}}\\
&\quad = \frac{\displaystyle\sum_{k=1}^{i-j+1} (-1)^{k+1} f_{kj} \prod_{q\neq k}^{i-j+1} a_q^{j+1} \prod_{\substack{m>n\\ m,n\neq k}}^{i-j+1} (a_m - a_n)}{\displaystyle\prod_{k=1}^{i-j+1} a_k^j \prod_{m>n}^{i-j+1} (a_m - a_n)} ,\\
&y(t_0 + h) = y_0 + h \sum_{k=1}^{i+1} f_{k-1}/k ,\\
&\text{where } j = 1, 2, \dots i;\ k = 1, 2, \dots i-j+1;\ i-j+1 > \dots > j+1 > j .
\end{aligned}\right\} \tag{A.12}$$

The process (A.12) requires the $[1 + i(i + 1)/2]$ th derivative evaluations to obtain the $(i + 1)$th degree numerical solution.

Various formulas obtained by the process (A.12) are listed in table 1 and the constants in these formulas are selected to

$$a_k = 1/k \,, \quad (k = 1, 2, \dots i)\,. \tag{A.13}$$

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