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NUMERICAL METHOD FOR SPACE HARMONIC WAVES IN POLYPHASE INDUCTION MOTORS

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A numerical method is proposed to solve the problem of space harmonic waves in polyphase induction motors. Results are compared with those of other numerical methods.

Notation

A	= parameter in numerical integration formulas
$C[v_m t]$	= commutation matrix of the 19th space harmonic wave
$G[v_m t]$	= $(\partial/\partial t)\, L[v_m t]\,(1/v_m)$, torque matrix
G^c	= transformed torque matrix
h	= stepwidth (sec)
$I[t]$	= $\{i_{sp}, i_{sn}, i_{rp}, i_{rn}\}$, current vector
$I[0]$	= initial current vector
$I^c[t]$	= $\{i^c_{sp}, i^c_{sn}, i^c_{rp}, i^c_{rn}\}$, transformed current vector
I_4	= unit matrix of order 4
j	= $\sqrt{-1}$, imaginary unit
k	= positive integer
$L[v_m t]$	= inductance matrix
L^c	= transformed inductance matrix
L_s, L_r	refer to the stator and rotor self-inductances, respectively
M_1, M_{19}	refer to the fundamental and 19th space harmonic wave mutual inductances, respectively
$P[t]$	= state transition matrix
p	= number of pole pairs
R	= $\lceil R_s, R_s, R_r, R_r \rfloor$, resistance matrix
R_s, R_r	refer to the stator and rotor resistances, respectively
$S[t]$	= $-L^{-1}[v_m t]\,(R + v_m G[v_m t])$, coefficient matrix of the differential state equation
S^c	= $-(L^c)^{-1}(R + v_m G^c)$, coefficient matrix of the linear differential state equation
T	= torque $(N-m)$
t	= time (sec)
$U[t]$	= $L^{-1}[v_m t]\, V[t]$, input vector of the differential state equation

$U^c[t]$ = $(L^c)^{-1} V^c[t]$, input vector of the linear differential state equation
$V[t]$ = $\{V_s \exp(jv_s t), V_s^* \exp(-jv_s t), 0, 0\}$, voltage vector
$V^c[t]$ = $C^*[v_m t] V[t]$, transformed voltage vector
V_s = amplitude of the stator impressed voltage
v_m = mechanical angular velocity transformed into electrical angular velocity (rad/sec)
v_s = angular velocity of the impressed voltage source
$Z[v_m t]$ = $R + v_m G[v_m t] + L[v_m t]$ (d/dt), impedance matrix
Z^c = $C^*[v_m t] Z[v_m t] C[v_m t]$ (or = $R + v_m G^c + L^c$ d/dt), transformed impedance matrix

Superscripts *, t, -1 and c refer respectively to complex conjugate matrix, transposed matrix, inverse matrix and matrix transformed by commutation matrix $C[v_m t]$.

Subscripts a, b, c refer to the stator a-phase, b-phase, c-phase quantities; s, r refer to the stator and rotor; and p, n refer to the positive and negative phase quantities, respectively.

1. Introduction

The accurate estimation of the torque produced by electric motors is important for the imporvement of various mechanical equipment driven by them. However, magnetic saturation and space harmonic wave effects make the accurate computation of the torque a difficult problem. Of all types of electric motors the polyphase induction motor is by far the most popular and the most widely used machine. Its space harmonic waves, however, produce abnormal torques and noise [1]–[5].

A numerical method that does not take space harmonic waves into account is given in [6]. However, if this method is applied to the problem of space harmonic waves in a polyphase induction motor, a very small stepwidth must be selected to obtain accurate results in spite of its time consuming nature. The reason is that the fundamental equations of polyphase induction motors including space harmonic waves consist of linear simultaneous differential equations with rapidly varying periodic coefficients. The purpose of this paper is to put forth a simple numerical method of effectively solving these equations with periodic coefficients. The solution obtained by the method takes into account exactly the space harmonic waves and, as a result, describes well the transient and steady state characteristics of polyphase induction motors. Furthermore, the method improves and generalizes the computational procedure described in [6].

The fundamental equations of polyphase induction motors considering space harmonic waves are introduced in terms of complex objects in section 2, and additional details are given in [5].

Section 3 describes the numerical method, where we at first approximate varying coefficients by keeping them constant over small time intervals and perform a direct integration of the fundamental differential equations over the above intervals to obtain analytical solutions.

Section 4 describes numerical results computed by the method of this paper. Comparison is made with various numerical schemes (see appendix 2).

2. Mathematical model

The linear simultaneous differential equations of polyphase induction motors are conveniently represented in matrix notation.

They involve the voltage vector $V[t]$ (t denotes the time), the impedance matrix $\boldsymbol{Z}[v_m t]$ which is a function of the time t and the mechanical angular velocity v_m, and the current vector $\boldsymbol{I}[t]$. The linear simultaneous differential equations with periodic coefficients are

$$V[t] = \boldsymbol{Z}[v_m t]\ \boldsymbol{I}[t]\ . \tag{1}$$

Each element of the voltage vector $V[t]$ is a state voltage, with amplitude V_s and angular velocity v_s. Then the voltage vector $V[t]$ (column vector of order 4) is

$$V[t] = \{V_s \exp(jv_s t),\ V_s^* \exp(-jv_s t),\ 0,\ 0\}\ , \tag{2}$$

where the superscript * denotes the conjugate quantities, and j is the imaginary unit ($j = \sqrt{-1}$).

The impedance matrix $\boldsymbol{Z}[v_m t]$ (square matrix of order 4) is a linear combination of the resistance matrix $\boldsymbol{R}$, the inductance matrix $\boldsymbol{L}[v_m t]$ and the torque matrix $\boldsymbol{G}[v_m t]$ (which can be obtained by differentiating the matrix $\boldsymbol{L}[v_m t]$ with respect to the time t and dividing the result by the mechanical angular velocity v_m). This yields

$$\boldsymbol{Z}[v_m t] = \boldsymbol{R} + v_m \boldsymbol{G}[v_m t] + \boldsymbol{L}[v_m t]\ (\mathrm{d}/\mathrm{d}t)\ . \tag{3}$$

The resistance matrix $\boldsymbol{R}$ (diagonal matrix of order 4) depends on the stator resistiance R_s and the rotor resistance R_r as follows:

$$\boldsymbol{R} = \lfloor R_s, R_s, R_r, R_r \rfloor\ . \tag{4}$$

The inductance matrix $\boldsymbol{L}[v_m t]$ is classified into four cases by the relations of numbers of rotor phases and of pole pairs p as described in [5]. The numerical method in this paper is applicable to all four cases. In order to illustrate the essential characteristics of the method, only a special case is treated as an example, where only the 19th space harmonic wave is taken into account. This example, at first, is practically useful because the 19th space harmonic wave has a big contribution to abnormal torque and then is theoretically very interesting as shown in appendix 1. In this case the elements of $\boldsymbol{L}[v_m t]$ are the stator self-inductance L_s, the rotor self-inductance L_r, the mutual inductance of the fundamental wave M_1 and the mutual inductance of 19th space harmonic wave M_{19}. The inductance matrix $\boldsymbol{L}[v_m t]$ (square matrix of order 4) is

$$\boldsymbol{L}[v_m t] = \begin{bmatrix} L_s & 0 & M_1 \exp(jv_m t) & M_{19} \exp(j19v_m t) \\ 0 & L_s & M_{19} \exp(-j19v_m t) & M_1 \exp(-jv_m t) \\ M_1 \exp(-jv_m t) & M_{19} \exp(j19v_m t) & L_r & 0 \\ M_{19} \exp(-j19v_m t) & M_1 \exp(jv_m t) & 0 & L_r \end{bmatrix} \tag{5}$$

The torque matrix $\boldsymbol{G}[v_m t]$ (square matrix of order 4) is given by

$$G[v_m t] = (\mathrm{d}/\mathrm{d}t)\, L[v_m t]\, (1/v_m)\,. \tag{6}$$

The current vector $I[t]$ (column vector of order 4) consists of the stator positive phase current i_{sp}, the stator negative phase current i_{sn}, the rotor positive phase current i_{rp} and the rotor negative phase current i_{rn}, viz.

$$I[t] = \{i_{sp}, i_{sn}, i_{rp}, i_{rn}\}\,. \tag{7}$$

The torque T is expressed in terms of the current vector $I[t]$, the torque matrix $G[v_m t]$ and the number of pole pairs p:

$$T = \mathcal{R}\{(p/2)(I^*[t])^{\mathrm{t}}\, G[v_m t]\, I[t]\}\,, \tag{8}$$

where $\mathcal{R}\{\cdot\}$ denotes the real part of $\{\cdot\}$, and $I^*[t]$ is the complex conjugate of $I[t]$.

For mathematical convenience the complex stator currents i_a, i_b, i_c are defined by the following equation, whose real parts give the original coordinate stator currents (in three-phase stator circuits):

$$\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-j2\pi/3) & \exp(-j4\pi/3) \\ 1 & \exp(-j4\pi/3) & \exp(-j2\pi/3) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ i_{sp} \\ i_{sn} \end{bmatrix}, \tag{9}$$

where the stator phases are represented by suffixes a, b and c.

3. Numerical method of solution

By means of eq. (3) it is possible to write eq. (1) as

$$(\mathrm{d}/\mathrm{d}t)\, I[t] = S[t] I[t] + U[t]\,, \tag{10}$$

where $S[t]$ is a square matrix of order 4 and $U[t]$ is a column vector of order 4 which are given by

$$S[t] = -L^{-1}[v_m t]\,(R + v_m G[v_m t])\,, \tag{11}$$

$$U[t] = L^{-1}[v_m t]\; V[t]\,. \tag{12}$$

A formal solution of eq. (10) can be obtained as

$$I[t] = P[t]\, I[0] + P[t] \int_0^t P^{-1}[r]\, U[r]\, \mathrm{d}r\,, \tag{13}$$

where the matrices $I[0]$ and $P[t]$ are respectively the initial value of the current vector (column vector of order 4) and the state transition matrix (square matrix of order 4). The state transition matrix $P[t]$ must satisfy the following equation

$$(\mathrm{d}/\mathrm{d}t)P[t] = S[t]\,P[t] \,. \tag{14}$$

The initial conditions of eq. (14) are

$$P[0] = I_4, \tag{15}$$

the unit matrix of order 4.

The state transition matrix $P[t]$ is of paramount importance in the solution of eq. (10). A satisfactory solution of eq. (10) depends on finding efficient methods of determining the state transition matrix $P[t]$. Various numerical methods have been proposed to find $P[t]$ (see [7]).

As the principal purpose of this paper is to solve eq. (10) by the simplest numerical method. It is assumed that the elements of the matrices $S[t]$ and $U[t]$ of eq. (10) keep constant values in a small time interval h. Then eq. (10) can be solved as follows:

$$I[t+h] = \exp(S[t+Ah]h)\,I[t] - \langle I_4 - \exp(S[t+Ah]h)\rangle\, S^{-1}[t+Ah]\; U[t+Ah] \,, \tag{16}$$

where the value for the parameter A will be given later.

A further approximation is applied to the matrix exponential function $\exp(S[t+Ah]h)$, namely

$$\exp(S[t+Ah]h) = \langle I_4 - (h/2)S[t+Ah]\rangle^{-1}\langle I_4 + (h/2)S[t+Ah]\rangle \,. \tag{17}$$

The state transition matrix $P[t+h]$ is approximated by the central difference method as follows:

$$P[t+h] = \langle I_4 - (h/2)S[t+Ah]\rangle^{-1}\langle I_4 + (h/2)S[t+Ah]\rangle\, P[t] \,. \tag{18}$$

To determine the value of the parameter A, eq. (18) is rewritten by using a Taylor series expansions of the matrices $S[t+Ah]$ and $\langle I_4 - (h/2)S[t+Ah]\rangle^{-1}\langle I_4 + (h/2)S[t+Ah]\rangle$. These Taylor series expansions are

$$S[t+Ah] = S[t] + Ah\,\mathrm{d}S[t]/\mathrm{d}t + (Ah)^2(1/2)\mathrm{d}^2S[t]/\mathrm{d}t^2$$
$$+ \ldots + (Ah)^k(1/k!)\,\mathrm{d}^kS[t]/\mathrm{d}t^k \tag{19}$$

and

$$\langle I_4 - (h/2)S[t+Ah]\rangle^{-1}\langle I_4 + (h/2)S[t+Ah]\rangle = I_4 + hS[t+Ah]$$
$$+ (h^2/2)S^2[t+Ah] + \ldots + (h^k/2^{k-1})S^k[t+Ah] \,. \tag{20}$$

Therefore, eq. (18) can be written as follows:

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + 2A\, dS[t]/dt] + (h^3/6)[1.5S^3[t] + 3A^2 d^2S[t]/dt^2 + 3AS[t]\, dS[t]/dt + 3A(dS[t]/dt)S[t]] + \ldots \rangle P[t] \,. \tag{21}$$

A rigorous Taylor series expansion of the state transition matrix $P[t+h]$ is described in [7]. It is

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + dS[t]/dt] + (h^3/6)[S^3[t] + d^2S[t]/dt^2 + S[t]\, dS[t]/dt + 2(dS[t]/dt)S[t]] + \ldots \rangle P[t] \,. \tag{22}$$

Since the approximate state transition matrix eq. (21) coincides with the rigorous expansion eq. (22) up to the first three terms if $A = 1/2$, we shall select

$$A = 1/2 \tag{23}$$

as most suitable.

To summarise this section, the formula for the linear simultaneous differential equations with periodic coefficients is given as

$$I[t+h] = \langle I_4 - (h/2)S[t+Ah]\rangle^{-1} \langle [I_4 + (h/2)S[t+Ah]]\; I[t] + hU[t+Ah]\rangle \,. \tag{24}$$

If the parameter A in eq. (24) is 1/2, then the approximate state transition matrix eq. (21) coincides with the rigorous expansion eq. (22) in the first three terms. This method is called the "improved central difference method". If the parameter A in eq. (24) is 1, then this method is the same as described in [6] (the central difference method), and the approximate state transition matrix eq. (21) matches the first two terms of the rigorous expansion eq. (22). In the case of linear simultaneous differential equations with constant coefficients, the formula denoted by eq. (24) ($A = 1/2$) is reduced to the central difference method.

4. Numerical solution

The various parameters of a motor used in the numerical examples are listed in table 1.

When we consider only the 19th space harmonic wave, the fundamental equations can be reduced to a set of linear differential equations with constant coefficients, as shown in appendix 1. To solve the coupled linear differential equations with constant coefficients, the Padé approximation method is known to be quite effective, although this method is not applicable to general problems of the polyphase induction motor including other space harmonic waves. Therefore, we obtain most accurate values for theoretical comparison by this method with very small stepwidth.

Eq. (1) is numerically solved by (a) the improved central difference method, (b) the trapezoidal rule, (c) the central difference method and (d) the Padé approximation method ($h = 0.000001$). A description of these methods is given in appendix 2.

The results of the numerical solution without considering the 19th space harmonic waves are

Table 1
Various constants of the calculated motor

Voltage	$V_s = \sqrt{2/3}\ 200$	(v)
Currents	initial currents are all zero	
Angular velocities	$v_s = 100\pi$	(rad/sec)
	$v_m = 90\pi$	(rad/sec)
Resistances	$R_s = R_r = 5$	(Ω)
	$L_s = L_r = 0.318\,31$	(H)
Inductances	$M_1 = 0.302\,39$	(H)
	$M_{19} = 0.302\,39/(19 \times 19)$	(H)

shown in fig. 1. Although fairly good results may be obtained by any numerical method by using a small stepwidth ($h = 0.000\,01$ or 0.0001), it is obvious that the improved central difference method is one of the most effective numerical methods.

When the 19th space harmonic wave is taken into account, the same conclusions are obtained for eq. (1). Results are shown in fig. 2.

The state transition matrices of the rigorous Taylor series method eq. (22), the improved central difference method, the trapezoidal rule and the central difference method are shown in table 2, which shows that the state transition matrix of the central difference method is inferior in accuracy to the other methods. The third term in the state transition matrix of the improved central difference method is somewhat more accurate than the third term of the trapezoidal rule. Furthermore, the trapezoidal rule needs more terms such as $\boldsymbol{S}[t]$ or $\boldsymbol{S}[t+h]$ and $\boldsymbol{U}[t]$ or $\boldsymbol{U}[t+h]$. Therefore, the improved central difference method is superior to the others.

Some examples of numerical solutions eqs. (1) and (8) computed by the improved central difference method are shown in fig. 3.

5. Conclusion

This paper has proposed a simple and effective method applicable to the problem of space harmonic waves in polyphase induction motors. This method is superior in accuracy to the trapezodial rule in spite of its simpler algorithmic form. A stepwidth can be chosen here that is about 20 times as large as the one necessary in the method reported in [6]. It is an improvement on the method reported in [6].

For further study the author plans to work out another numerical method for space harmonic waves in polyphase induction motors, taking into account the full system of mechanical equations.

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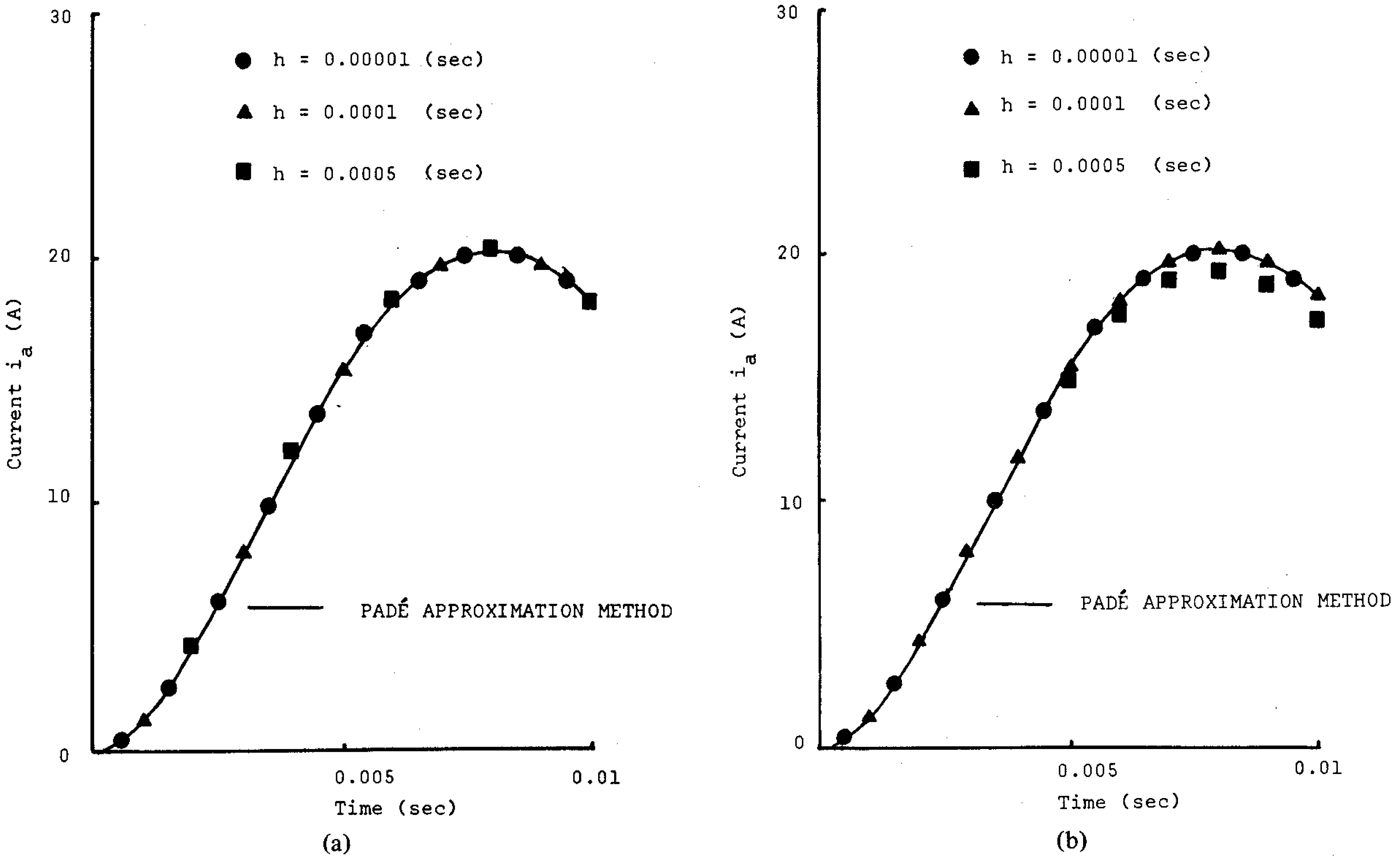


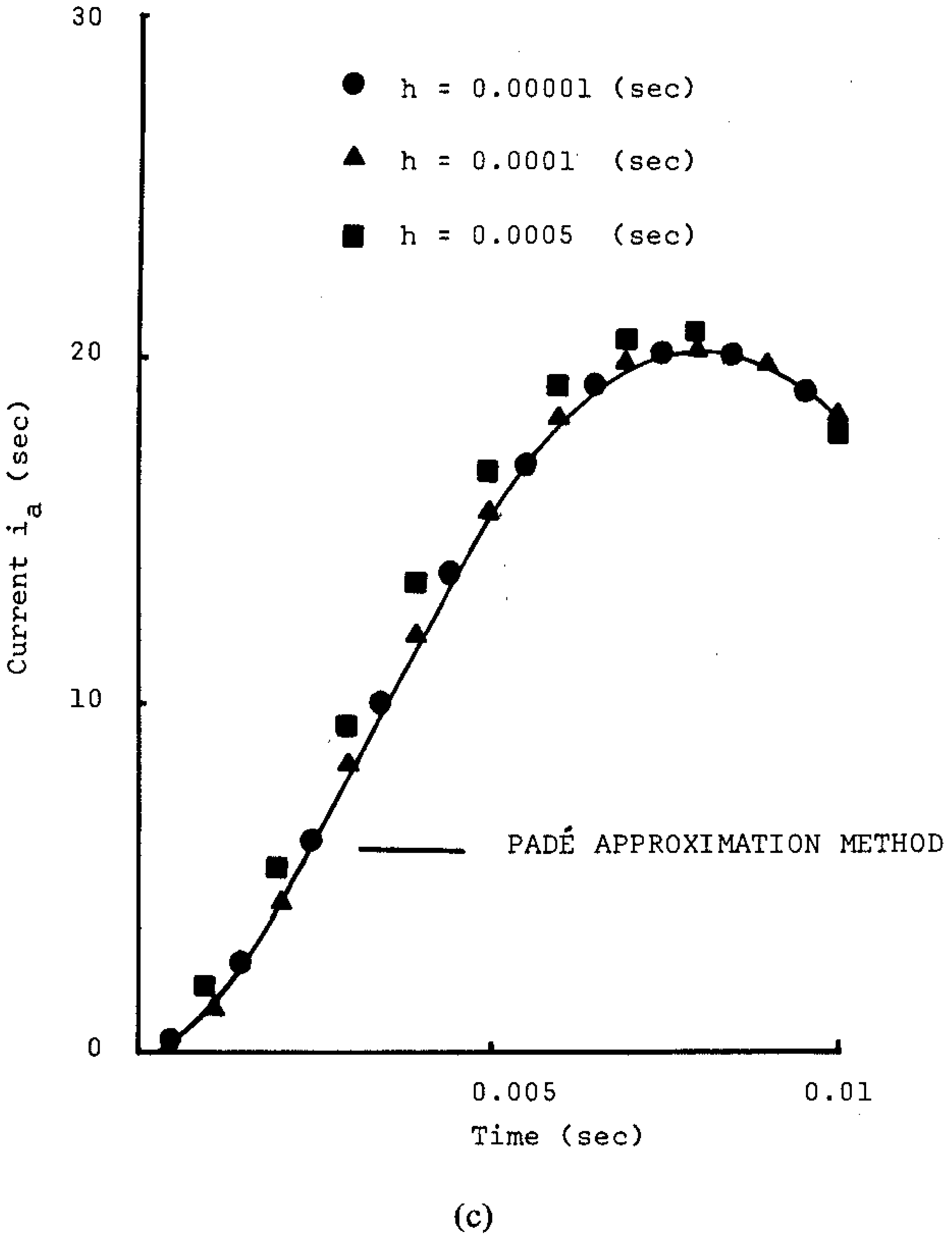
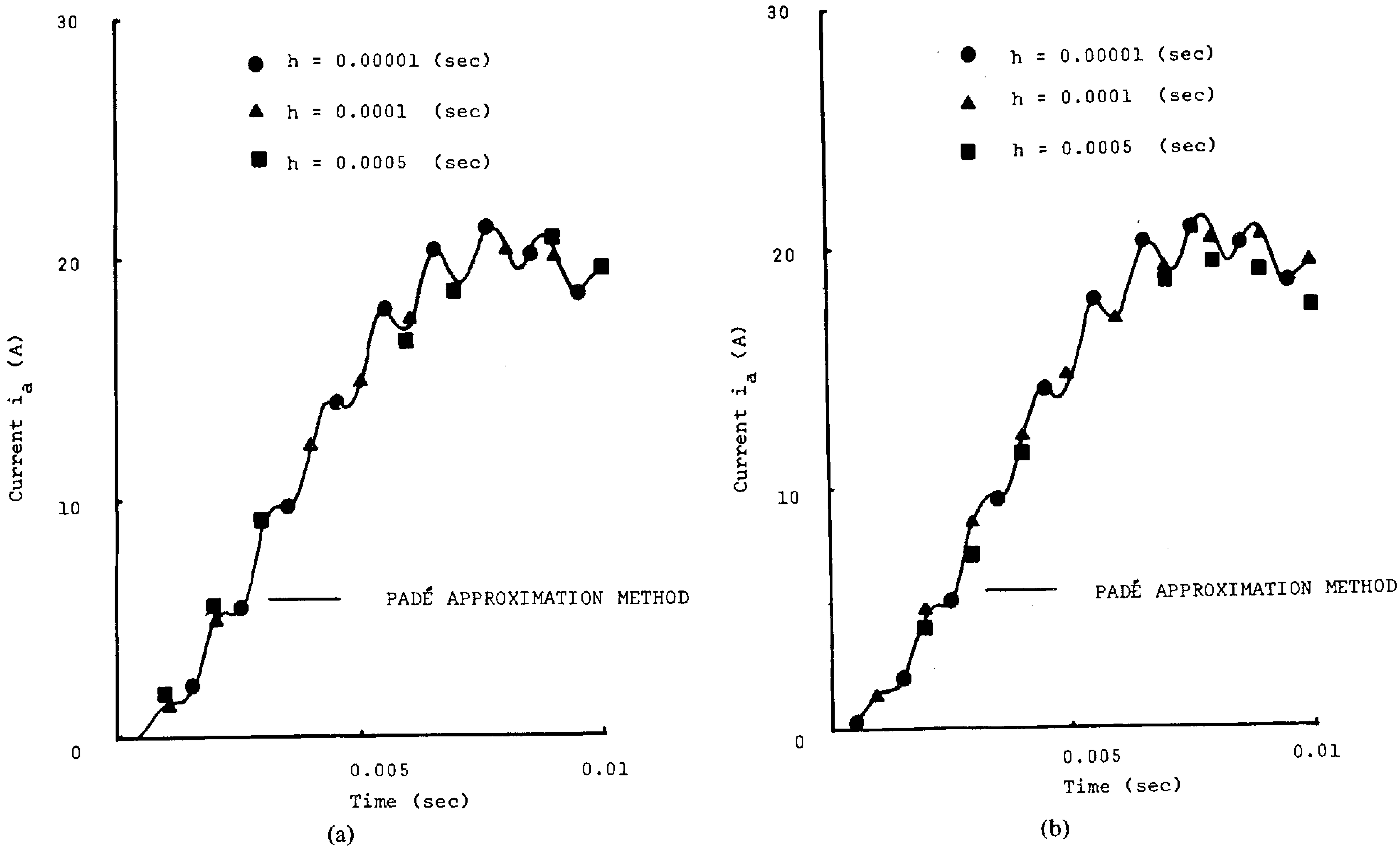


Fig. 1. Comparison of the results not considering the 19th space harmonic wave computed by various numerical methods: (a) Improved central difference method, (b) Trapezoidal rule, (c) Central difference method.



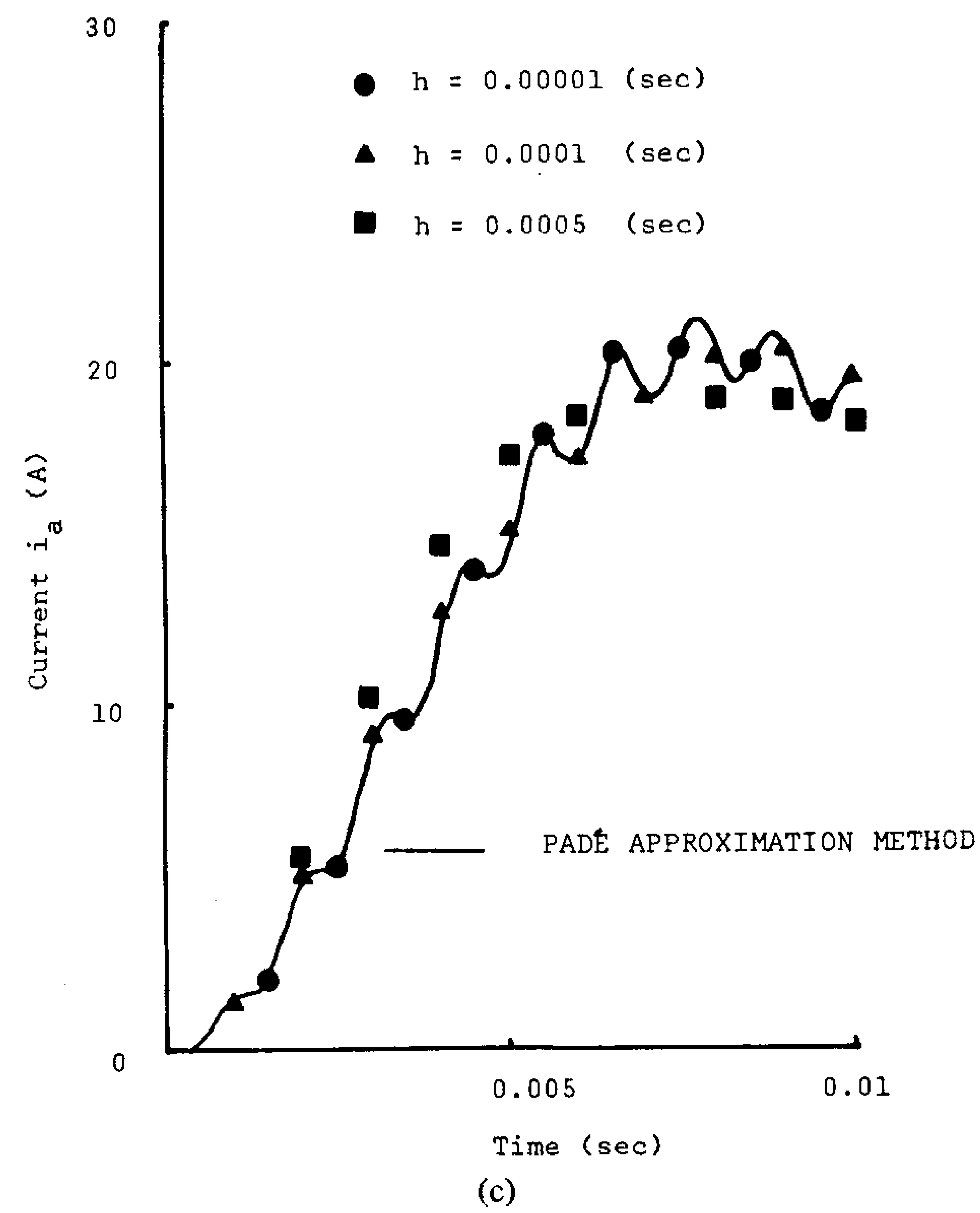
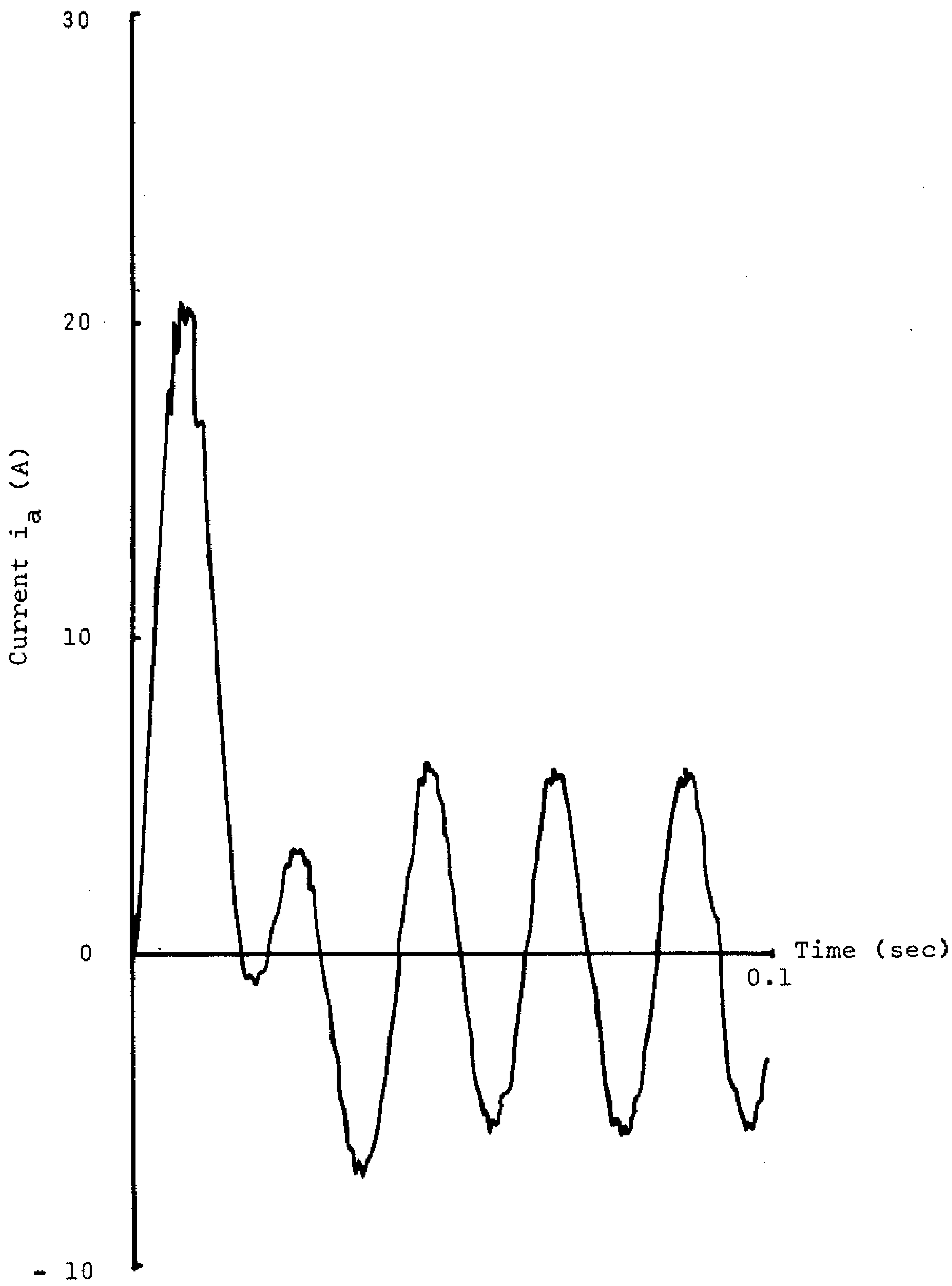
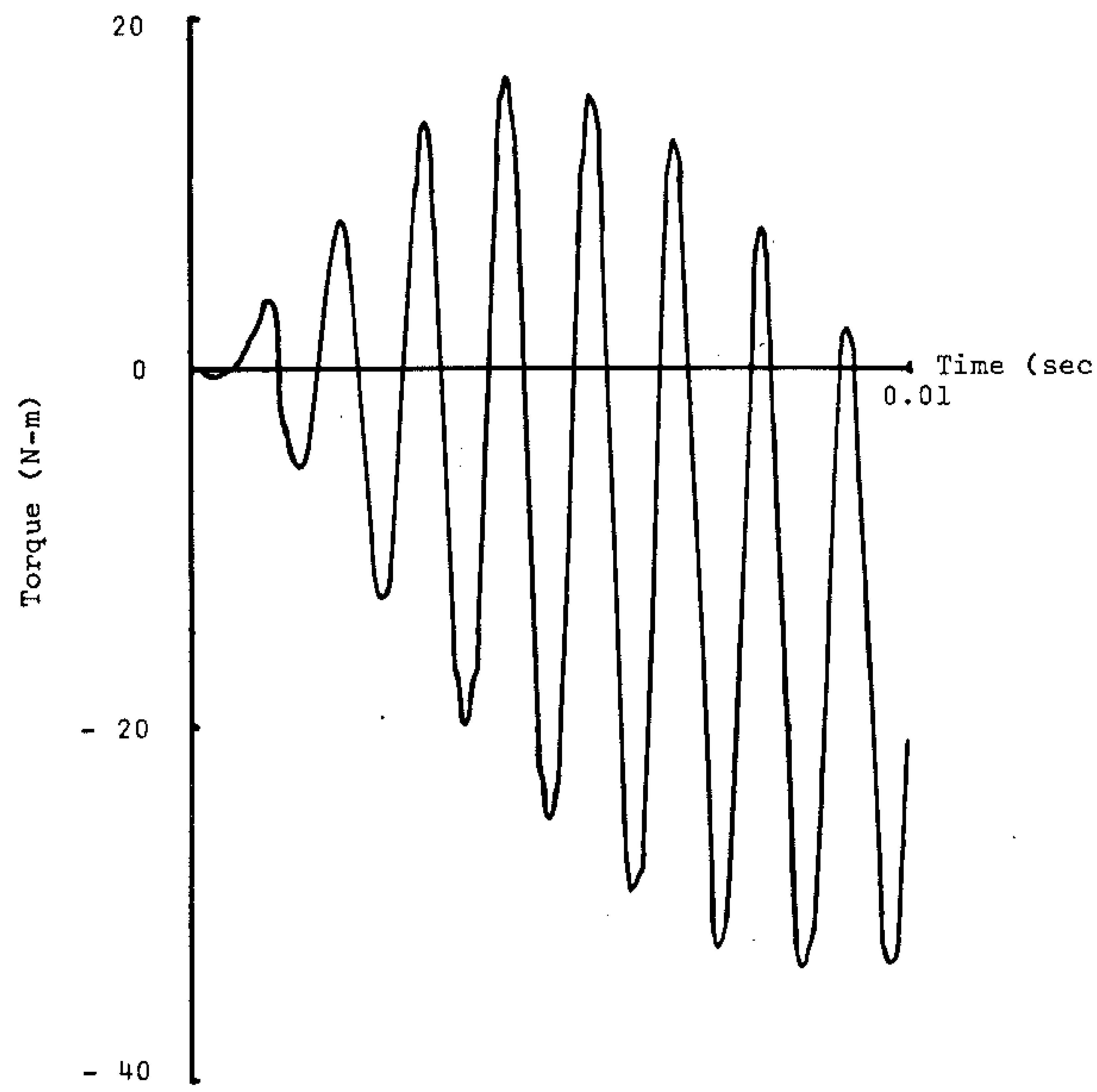


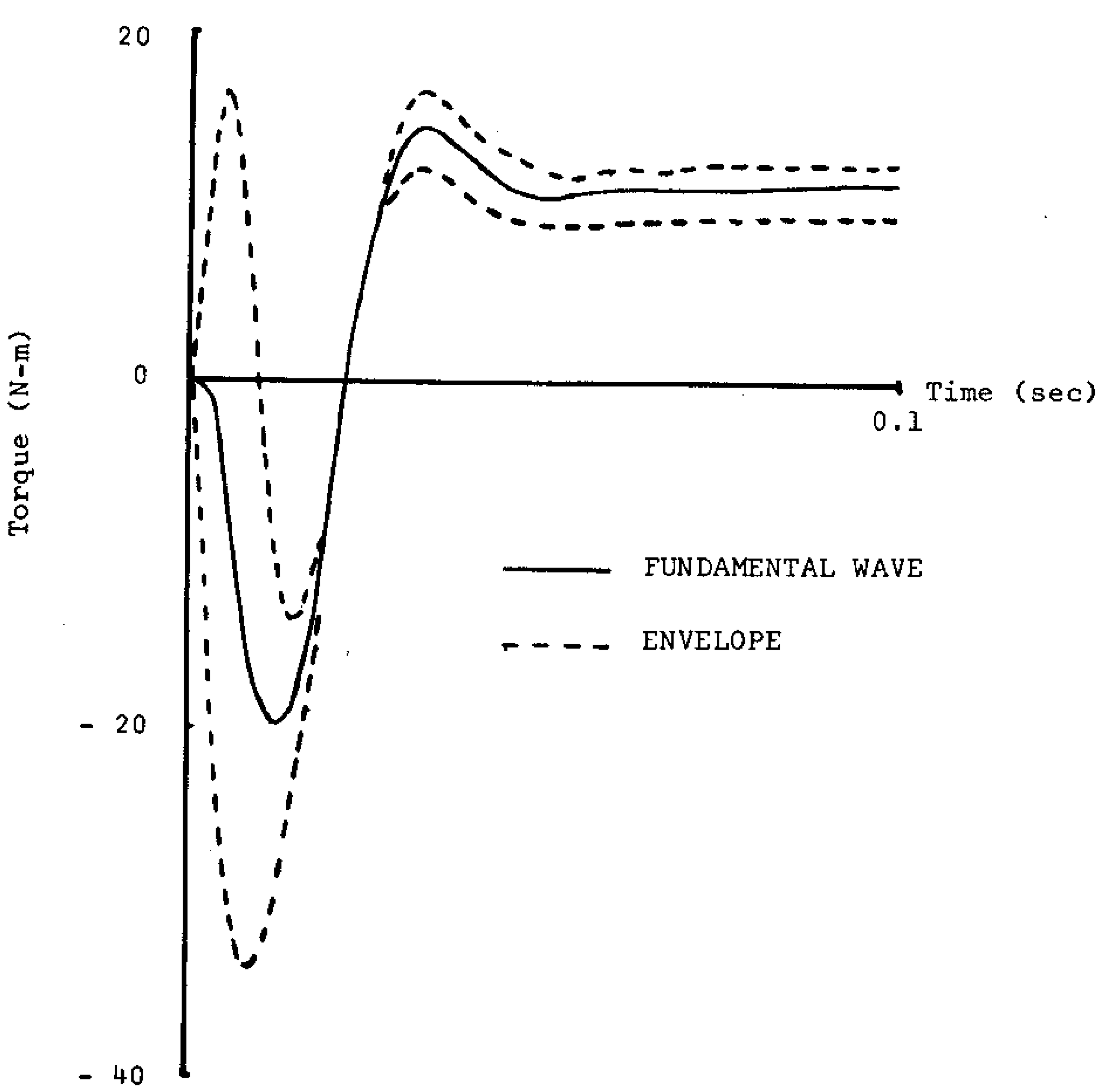
Fig. 2. Comparison of the results taking into account the 19th space harmonic wave: (a) Improved central difference method, (b) Tapezoidal rule, (c) Central difference method.



(a)



(b)



(c)

Fig. 3. Examples of the numerical solutions computed by the improved central difference method, using the stepwidth h = 0.0001: (a) Numerical solution of the stator current i_a, (b) Numerical solution of the torque, (c) Numerical solution of the torque (envelope in the figure shows the region of the torque vibration due to the 19th space harmonic wave).

Table 2
Comparison of the approximate and the rigorous state transition matrix

Rigorous Taylor series expansion

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + dS[t]/dt] + (h^3/6)[S^3[t] + d^2S[t]/dt^2 + S[t]\, dS[t]/dt + 2(dS[t]/dt)S[t]] + \ldots \rangle P[t]$$

Improved central difference method

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + dS[t]/dt] + (h^3/6)[1.5S^3[t] + 0.75\, d^2S[t]/dt^2 + 1.5S[t]\, dS[t]/dt + 1.5(dS[t]/dt)S[t]] + \ldots \rangle P[t]$$

Trapezoidal rule

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + dS[t]/dt] + (h^3/6)[1.5S^3[t] + 1.5d^2S[t]/dt^2 + 1.5S[t]\, dS[t]/dt + 3(dS[t]/dt)S[t]] + \ldots \rangle P[t]$$

Central difference method

$$P[t+h] = \langle I_4 + hS[t] + (h^2/2)[S^2[t] + 2dS[t]/dt] + (h^3/6)[1.5S^3[t] + 3d^2S[t]/dt^2 + 3S[t]\, dS[t]/dt + 3(dS[t]/dt)S[t]] + \ldots \rangle P[t]$$

The practical computations given in this paper were carried out by using the computers FACOM 230–45S of Hosei University and HITAC 8800/8700 of Tokyo University.

Appendix 1. *Linearized mathematical model*

Eq. (1) in section 2 can be transformed into a linear system of simultaneous differential equations with constant coefficients by using the matrix $C[v_m t]$ (commutation matrix) for space harmonic waves. The commutation matrix $C[v_m t]$ (diagonal matrix of order 4) for the 19th space harmonic wave is

$$C[v_m t] = \lceil 1, \exp(-j20v_m t), \exp(-jv_m t), \exp(-j19v_m t)\rfloor . \quad \text{(A.1)}$$

The linear simultaneous differential equations with constant coefficients are written as follows:

$$V^c[t] = Z^c I^c[t] . \quad \text{(A.2)}$$

The transformed voltage vector (column vector of order 4) $V^c[t] = C^*[v_m t]\, V[t]$ is

$$V^c[t] = \{V_s \exp(jv_s t), V_s^* \exp(-jv_s t + j20v_m t), 0, 0\}. \quad \text{(A.3)}$$

The transformed impedance matrix $Z^c = C^*[v_m t]\, Z[v_m t]\, C[v_m t]$ (square matrix of order 4) consists of the resistance matrix R the transformed inductance matrix L^c and the transformed torque matrix $\boldsymbol{G}^c$, namely

$$\boldsymbol{Z}^c = \boldsymbol{R} + v_m \boldsymbol{G}^c + \boldsymbol{L}^c (\mathrm{d}/\mathrm{d}t)\,, \tag{A.4}$$

where the resistance matrix $\boldsymbol{R}$ is given in eq. (4), and the transformed inductance matrix $\boldsymbol{L}^c$ (square matrix of order 4) is

$$\boldsymbol{L}^c = \begin{bmatrix} L_s & 0 & M_1 & M_{19} \\ 0 & L_s & M_{19} & M_1 \\ M_1 & M_{19} & L_r & 0 \\ M_{19} & M_1 & 0 & L_r \end{bmatrix}. \tag{A.5}$$

The transformed torque matrix $\boldsymbol{G}^c$ (square matrix of order 4) is

$$\boldsymbol{G}^c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -j20L_s & -j20M_{19} & -j20M_1 \\ -jM_1 & -jM_{19} & -jL_r & 0 \\ -j19M_{19} & -j19M_1 & 0 & -j19L_r \end{bmatrix}. \tag{A.6}$$

As each current in the current vector $\boldsymbol{I}[t]$ is an unknown quantity which must be computed, the transformed current vector $\boldsymbol{I}^c[t] = \boldsymbol{C}^*[v_m t]\,\boldsymbol{I}[t]$ is represented by the new stator positive phase current i^c_{sp}, the new stator negative phase current i^c_{sn}, the new rotor positive phase current i^c_{rp} and the new rotor negative phase current i^c_{rn}. The transformed current vector $\boldsymbol{I}^c[t]$ (column vector of order 4) is

$$\boldsymbol{I}^c[t] = \{i^c_{sp}, i^c_{sn}, i^c_{rp}, i^c_{rn}\}\,. \tag{A.7}$$

The torque $\boldsymbol{T}$ is represented by the transformed current vector $\boldsymbol{I}^c[t]$, the transformed torque matrix $\boldsymbol{G}^c$ and the number of pole pairs p. Namely, the torque $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \mathcal{R}\{p(\boldsymbol{I}^{c*}[t])^{\mathrm{t}}\, \boldsymbol{G}^c \boldsymbol{I}^c[t]\}\,, \tag{A.8}$$

$\{\cdot\}$ = real part of $\{\cdot\}$.

Of course, the real part of eq. (A.8) equals the real part of eq. (8).

The original coordinate stator currents (in three phase circuits) are derived as the real part of the following equation:

$$\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp(-j2\pi/3) & \exp(-j4\pi/3) \\ 1 & \exp(-j4\pi/3) & \exp(-j2\pi/3) \end{bmatrix} \begin{bmatrix} 0 \\ i_{sp}^c \\ i_{sn}^c \exp(-j20v_m t) \end{bmatrix} . \tag{A.9}$$

Appendix 2. *Other numerical methods*

1) Trapezoidal rule: Application of the trapezoidal integral formula to eq. (10) in section 3 yields

$$I[t+h] - I[t] = (h/2)\{S[t+h]\, I[t+h] + S[t]\, I[t] + U[t+h] + U[t]\} . \tag{A.10}$$

Therefore, the following numerical solution of eq. (10) can be obtained:

$$I[t+h] = \langle I_4 - (h/2)S[t+h]\rangle^{-1} \langle I_4 + (h/2)S[t]\rangle\, I[t]$$
$$+ \langle I_4 - (h/2)S[t+h]\rangle^{-1} (h/2) \langle U[t+h] + U[t]\rangle . \tag{A.11}$$

In this case the state transition matrix $P[t+h]$ is approximated as follows:

$$P[t+h] = \langle I_4 - (h/2)S[t+h]\rangle^{-1} \langle I_4 + (h/2)S[t]\rangle\, P[t]$$
$$= \langle I_4 + hS[t] + (h^2/2)[S^2[t] + \mathrm{d}S[t]/\mathrm{d}t] \tag{A.12}$$
$$+ (h^3/6)[1.5S^3[t] + 1.5\, \mathrm{d}^2S[t]/\mathrm{d}t^2 + 3(\mathrm{d}S[t]/\mathrm{d}t)S[t] + 1.5S[t]\, \mathrm{d}S[t]/\mathrm{d}t] + \dots \rangle P[t] .$$

2) Padé approximation method: From eq. (16) the numerical solution of eq. (A.2) can be obtained as follows:

$$I^c[t+h] = \exp(S^c h)\, I^c[t] - \langle I_4 - \exp(S^c h)\rangle\, S^{c-1}\, U^c[t+0.5h] , \tag{A.13}$$

where the parameter A in eq. (16) is conveniently selected to be 1/2, and the matrices in eq. (A.13) are

$$S^c = -L^{c-1}(R + v_m G^c) , \tag{A.14}$$

$$U^c[t] = L^{c-1}\, V^c[t] . \tag{A.15}$$

The matrix exponential function $\exp(S^c h)$ in eq. (A.13) is approximated by a Padé approximation [8], that is

$$\exp(S^c h) = \langle 12 - 6S^c h + (S^c h)^2 \rangle^{-1} \langle 12 + 6S^c h + (S^c h)^2 \rangle$$
$$= I_4 + hS^c + (hS^c)^2/2 + (hS^c)^3/6 + (hS^c)^4/24 + \ldots . \quad \text{(A.16)}$$

As the right-hand term of eq. (A.16) is a relatively good approximation of the matrix exponential function in eq. (A.13), it can be expected that the numerical solutions computed by the method of eq. (A.13) using eq. (A.16) will yield fairly good results.

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