A meaningful post-processing method based on a locally orthogonal discretization

Y. Saito, S. Ikeguchi, and S. Hayano
College of Engineering, Hosei University, 3-7-2 Kajinocho Koganei, Tokyo 184, Japan

A locally orthogonal property between the side of a Delaunay triangle and the edge of a Voronoi polygon is applied to the post-processing scheme. Our post-processing scheme explicitly satisfies a governing equation in the electromagnetic fields so that it removes the discontinuities appearing at the edges of the first-order triangular finite element. Furthermore, when the vector obtained by our post-processing scheme is used as a starting vector for the iterative matrix inversions, e.g., the successive overrelaxation and conjugate gradient methods, then covergence to a correct solution vector is extremely accelerated in our test examples.

I. INTRODUCTION

Recently we proposed a new method which uses a single potential based on a geometrical dual property of locally orthogonal discretization. A locally orthogonal property arises from the following fact that each side of a Delaunay triangle is perpendicular to the corresponding edge of a Voronoi polygon. This leads to two independent systems, i.e., the Delaunay and Voronoi systems. In a previous paper, we applied this locally orthogonal property to the complementary variational principles, and succeeded in evaluating the highly improved functional as well as potential with low computational cost.[1-3]

In this paper a locally orthogonal property between the side of a Delaunay triangle and the edge of a Voronoi polygon is applied to the post-processing scheme. Our post-processing scheme explicitly satisfies a governing equation in the electromagnetic fields so that it removes the discontinuities in a field appearing at the edges of the first-order finite element. Furthermore, when the vector obtained by our post-processing scheme is used as a starting vector for an iterative matrix inversion, then convergence to a correct solution vector is extremely accelerated.

II. THE LOCALLY ORTHOGONAL DISCRETIZATION METHOD

A. Assumptions

Numerous problems in electrical engineering are reduced to solve Poisson's equation in two dimensions:

$$\lambda \frac{\partial^2\phi}{\partial x^2} + \lambda \frac{\partial^2\phi}{\partial y^2} = -\sigma, \tag{1}$$

where λ is a parameter depending on the medium; ϕ is the scalar or axial component of vector; and σ is a source density. At the boundary of region 0 and 1, the following boundary conditions are assumed:

$$\left.\frac{\partial\phi}{\partial x}\right|_{\text{Region 0}} = \left.\frac{\partial\phi}{\partial x}\right|_{\text{Region 1}}, \tag{2}$$

$$\lambda_0\left.\frac{\partial\phi}{\partial y}\right|_{\text{Region 0}} = \lambda_1\left.\frac{\partial\phi}{\partial y}\right|_{\text{Region 1}}. \tag{3}$$

When a problem region is discretized into triangular elements, then a triangle is generally divided into two or three isosceles triangles by taking the circumcenter as a vertex, as shown in Fig. 1. It is apparent that each of the edges of the triangle is perpendicularly intersected by a line connecting the circumcenters of the adjoint triangles. Thus, the lines *i-j* and 0-1 in Fig. 1 form a locally orthogonal system, and this relationship leads to the two independent trial functions for the primal and complementary functionals. Thereby, on the local coordinate system shown in Fig. 1, the solution of Eq. (1) is assumed to be evaluated from[1,2]

$$\lambda\frac{\partial^2\phi}{\partial x^2} = -\tfrac{1}{2}\sigma, \tag{4}$$

$$\lambda\frac{\partial^2\phi}{\partial y^2} = -\tfrac{1}{2}\sigma. \tag{5}$$

B. Functionals

On the local coordinate system shown in Fig. 1, the nodes *i*, *j* are located on the boundary between regions 0 and 1. This means that the rate of change $\partial\phi/\partial x$ is common to both regions 0 and 1 in Fig. 1. Thereby, the primal functional for Eqs. (2) and (4) is written by[1]

$$F(\phi) = \int \lambda \left(\frac{\partial\phi}{\partial x}\right)^2 dx\, dy - \int \phi\sigma\, dx\, dy, \tag{6}$$

where the integrations are carried out over the region enclosed by 0-*d*-*i*-*e*-1-*j*-0 in Fig. 1.

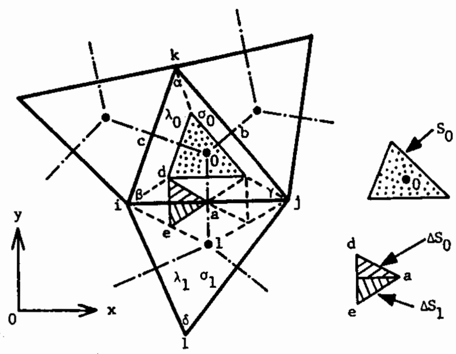

FIG. 1. An example of a locally orthogonal coordinate system, where nodes 0,1,*d*,*e* are located at the circumcenters of triangles *i-j-k*, *i-j-l*, *i-0-a*, and *i-1-a*, respectively.

 0021-8979/88/083369-03$02.40

On the other side, nodes 0,1 are located on the y axis in Fig. 1. In this case, the rate of change $\lambda(\partial\phi/\partial y)$ is common to both regions 0 and 1. This yields the complementary functional $G(\phi)$ for Eqs. (3) and (5) as[1,2]

$$G(\phi) = -\int \frac{1}{\lambda}\left(\lambda \frac{\partial \phi}{\partial y}\right)^2 dx\,dy + \int \hat{\phi}\sigma\, dx\,dy, \qquad (7)$$

where $\hat{\phi}$ denotes the prescribed value of potential ϕ at nodes 0,1; and the integrations are carried out over the region enclosed by 0-d-i-e-1-j-0 in Fig. 1.

C. Node equations

A simple Lagrangian interpolation between the nodes i and j in Fig. 1 yields a trial function for the primal functional as

$$\phi_p = \tfrac{1}{2}(\phi_i + \phi_j) + (\phi_i - \phi_j)(x/a), \qquad (8)$$

where a is the distance between nodes i and j. Equation (8) satisfies the boundary condition (2). Minimizing after introducing Eq. (8) into Eq. (6) yields the node equations. For example, the equation for node i is given by

$$\frac{\partial F(\phi_p)}{\partial \phi_i} = \left(\frac{\lambda_0}{2}\cot\alpha + \frac{\lambda_1}{2}\cot\delta\right) \times(\phi_i - \phi_j) - 4(\Delta S_0\sigma_0 + \Delta S_1\sigma_1) = 0, \qquad (9)$$

where the areas ΔS_0,ΔS_1 and angles α, δ are shown in Fig. 1. The other node equations for the primal system can be derived in much the same way as Eq. (9); a combination of these equations leads to a primal system.[1]

In order to satisfy condition (3), it is inevitable to use two different trial functions:

$$\phi_c = [(\lambda_0/b)\phi_0 + (\lambda_1/c)\phi_1]/[(\lambda_0/b) + (\lambda_1/c)] + [(\lambda_1/bc)(\phi_0 - \phi_1)]y/[(\lambda_0/b) + (\lambda_1/c)], \quad (10a)$$

$$\phi_c = [(\lambda_0/b)\phi_0 + (\lambda_1/c)\phi_1]/[(\lambda_0/b) + (\lambda_1/c)] + [(\lambda_0/bc)(\phi_0 - \phi_1)]y/[(\lambda_0/b) + (\lambda_1/c)], \quad (10b)$$

where Eqs. (10a) and (10b) are held within regions 0 and 1; b,c are the distances between nodes 0,a and nodes a,1 in Fig. 1, respectively. Maximization after introducing Eqs. (10a) and (10b) into Eq. (7) yields the node equations. For example, the equation for node 0 in Fig. 1 is given by

$$\frac{\partial G(\phi_c)}{\partial \phi_0} = (\phi_1 - \phi_0)/[(1/2\lambda_0)\cot\alpha + (1/2\lambda_1)\cot\delta] + 8\Delta S_0\sigma_0 = 0. \qquad (11)$$

The other node equations for the complementary system can be obtained in much the same way as Eq. (11); a combination of these node equations yields a complementary system.[1,2]

D. Application to post-processing

Let us assume that the node potentials ϕ_i,ϕ_j,ϕ_k,ϕ_l in Fig. 1 have been obtained by the first-order triangular finite elements method, then the primal system gives a potential ϕ_a located at a midpoint of edge i-j as a solution of

$$[(\lambda_0/2)\cot\alpha + (\lambda_1/2)\cot\delta](\phi_a - \phi_i) + [(\lambda_0/2)\cot\alpha + (\lambda_1/2)\cot\delta](\phi_a - \phi_j) - 2(\Delta S_0\sigma_0 + \Delta S_1\sigma_1) = 0, \qquad (12)$$

where the area ΔS_0, ΔS_1 are shown in Fig. 1. Similarly, the other potentials ϕ_b, ϕ_c on the edges of triangle i-j-k in Fig. 1 can be obtained in much the same way as Eq. (12). Also, complementary system gives a potential ϕ_0 in Fig. 1 as a solution of

$$\lambda_0[\tan\alpha(\phi_a - \phi_0) + \tan\beta(\phi_b - \phi_0) + \tan\gamma(\phi_c - \phi_0)] + S_0\sigma_0 = 0, \qquad (13)$$

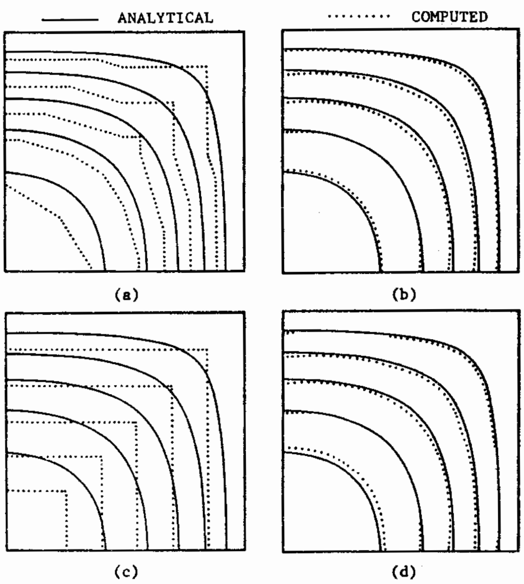

FIG. 2. Post-processing results. (a) Original magnetic fields, where the five node potentials are the boundary values and four node potentials are the solutions of FEM. (b) A post-processed result of (a), where the extra 280 node potentials have been added. (c) Original magnetic fields, where only one node potential is a solution of FEM and three node potentials are the boundary values. (d) A post-processed result of (c), where the extra 285 node potentials have been added. The solid and dotted lines are the analytical and computed values, respectively.

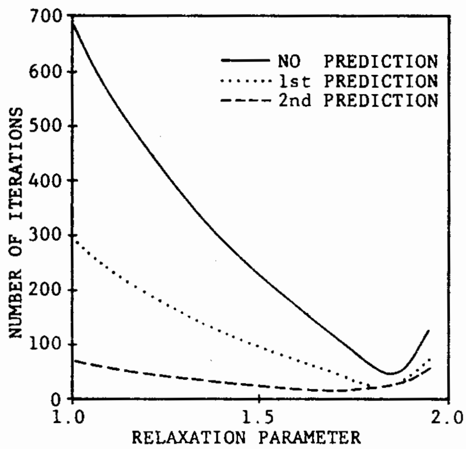

FIG. 3. A convergence property of the SOR method. No prediction: zero starting vector. First prediction: a starting vector was obtained from Fig. 2(d). Second prediction: a starting vector was obtained from Fig. 2(b).

where angles α,β,γ and area S_0 are shown in Fig. 1.

Thus, by means of the process shown in Eqs. (12) and (13), we can add any number of node potentials. Equation (12) satisfies one axial component but Eq. (13) satisfies both of the x and y components of Eq. (1) within second-order terms when solution ϕ is expanded into a Taylor series.[3]

E. Application to the iterative methods of solution

Iterative methods such as the SOR and CG methods are widely available to solve large and medium size problems. In these iterative methods, the number of iterations required to arrive at a correct solution greatly depends on the starting value. Our post-processing scheme yields the potentials which satisfy an original governing equation (1) so that a vector obtained by the process of Eqs. (12) and (13) may become a good starting vector for the iterative methods of solution.

F. Examples

The method is illustrated by applying to a magnetic field calculation of ferromagnetic material with square cross section.[1,4] Figure 2(a) shows the original magnetic fields computed by the first-order finite elements, where five node potentials are the boundary values and the remaining four node potentials are the solutions of FEM. Figure 2(b) shows the results of our post-processing, where 280 extra node potentials have been added. A further severe condition is shown in Fig. 2(c), where only one node potential is a solution of FEM and the other three node potentials are the boundary values. As shown in Fig. 2(d), the discontinuities in Fig. 2(c) were removed by our post-processing method. In Fig. 2(d), 285 extra node potentials were added by our post-processing scheme. In order to evaluate 256 node potentials, the results of Figs. 2(b) and 2(d) were used as the starting vectors for the iterative methods of solution. Figure 3 shows a convergence property of the SOR method and reveals that our post-processing scheme may be used for predicting the starting vector for iterative approaches.

III. CONCLUSION

In this paper we have proposed a new post-processing method based on the locally orthogonal discretizations. As a result, the extremely smooth fields may be obtained at the post-processing stage. Also, it has been shown that our post-processing scheme may be used for predicting the starting vector for iterative methods of solution.

[1]Y. Saito, Y. Kishino, S. Hayano, H. Nakamura, N. Tsuya, and Z. J. Cendes, IEEE Trans. Magn. **MAG-22**, 1057 (1986).
[2]Y. Saito, S. Hayano, and N. Tsuya, IEEE Trans. Magn. **MAG-21**, 2280 (1985).
[3]Y. Saito, S. Hayano, Y. Kishino, S. Ikeguchi, and K. Fukushima, IEE J. Magn. **MAG-86**, 112 (1986).
[4]P. Hammond and T. D. Tsiboukis, Proc. IEE **130**, Pt. A, 105 (1983).