Magnetic circuit analysis based on the finite elements

Generalization to the quasi-three-dimensional problem

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Abstract. As is well known, conventional magnetic circuit theory makes it possible to analyze the three-dimensional magnetic fields, even though its theoretical background is based on the engineering experiences. On the other side, finite elements have a firm theoretical background but it is too expensive to implement the three-dimensional analysis. In order to carry out the three-dimensional magnetic field analysis in a most efficient manner, we propose the theory of modern magnet circuit based on finite elements. This makes it possible to implement the three-dimensional magnetic field computation in a quite efficient manner. A simple example demonstrates not only the usefulness but also efficiency of our new methodology.

1. Introduction

Calculation of magnetic fields is one of the most important factors to design electromagnetic devices. Because of the nonlinear magnetization characteristic of iron, however, it is extremely difficult to obtain an analytical solution of the magnetic fields in electromagnetic devices. Thereby, there were only conventional design methods which based to a considerable extent on conventional magnetic circuit theory and engineering experiences [1]. This means that these design methods are insufficient to evaluate the exact dynamic performance of electromagnetic devices. On the other side, with the development of modern computers, numerical methods (e.g. the finite element method) became available to compute the magnetic fields in electromagnetic devices, taking into account the fully nonlinear magnetization characteristics of iron. The finite elements have a firm theoretical background but it is too expensive to implement the three-dimensional analysis [2].

In order to overcome these difficulties and to carry out the three-dimensional magnetic field analysis in a most efficient manner, we propose the theory of modern magnetic circuits (TMMC, in short) based on the finite elements. This makes it possible to implement the three-dimensional magnetic field computation in a quite efficient manner similar to those of the conventional magnetic circuit theory.

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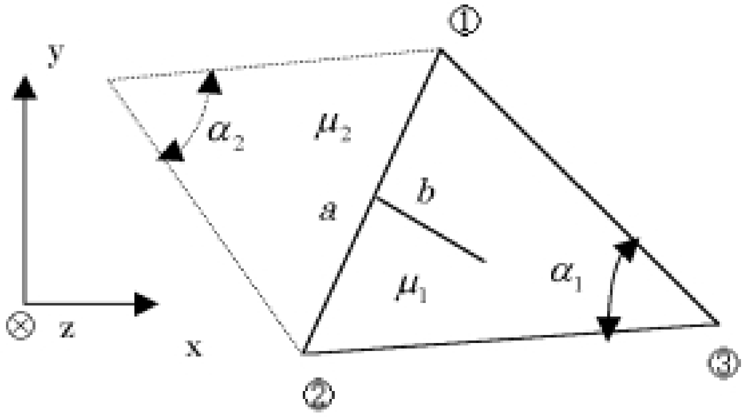

Fig. 1. 1st order triangular finite element.

2. The theory of modern magnetic circuits

2.1. 2D TMMC

When we consider a simple 2D problem carrying a uniform current density **J** in an axial direction. Governing equation in this problem becomes Poisson equation given by Eq. (1):

$$\frac{1}{\mu}\nabla^2 A = -J. \tag{1}$$

Where the vector potential $\mathbf{A}^*$ is expressed in terms of the position variables x and y:

$$\mathbf{A}^* = f(x, y). \tag{2}$$

in each of the subdivided finite elements, finite element solution can be obtained by minimizing a functional of Eq. (3).

$$F(A^*) = \frac{1}{2}\int_{\mathbf{s}} (\nabla\mathbf{A}^*)^2 ds - \int_s J\mathbf{A}^* ds. \tag{3}$$

Let us consider a first order base function to Eq. (2), then an approximately solution of Eq. (1) are determined by minimizing the functional Eq. (3). Applying Eq. (3) to one of the triangular elements shown in Fig. 1, then it is possible to obtain a parameter determined by the geometrical and medium parameters μ as

$$G_{1,2} = \frac{1}{2\mu_1}\cot\alpha_1\cdot \tag{4}$$

Furthermore, we denote that the lengths a and b are the distance between the nodes (1)–(2) and the distance of the perpendicular taken down from circumcenter of a triangle (1)(2)(3) to the side (1)–(2), respectively. This leads to a relationship:

$$G_{1,2} = \frac{1}{2\mu_1}\left(\frac{b}{\frac{a}{2}}\right) = \frac{b}{\mu_1 a}\cdot \tag{5}$$

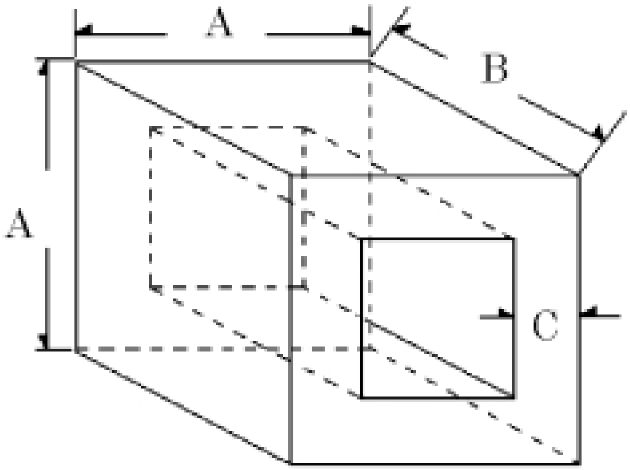

Fig. 2. Hollow square cubic.

Considering the relationship Eq. (5) leads to a magnetic circuit Eq. (6). Further, minimization of the functional of Eq. (3) along with the $16\mathrm{st}$ order base function makes it reduce the current density J as the nodal currents at the vertices in Fig. 2. Integration of the vector potential along with its direction becomes a magnetic flux so that a magnetic circuit equation of Eq. (1) based on 1st order finite elements can be derived as

$$G_{1,2}(A_1 - A_2) = \frac{b}{\mu_1 a}(A_1 - A_2) = \frac{b}{\mu_1 a}(A_1 - A_2) = R_{1,2}(\phi_1 - \phi_2), \tag{6}$$
$$\phi_1 = A_1 \cdot 1,\ \phi_2 = A_2 \cdot 1,\ R_{1,2} = \frac{b}{\mu_1 a \cdot 1},$$

where 1 denotes an unit length in the direction of vector potential, and

$$\frac{1}{3}\sum_{i=1}^{m} S_i J_i = \sum_{i=1}^{m} R_{1,f}(\phi_1 - \phi_i)\cdot \tag{7}$$

2.2. *Quasi-3D TMMC*

Let us consider a magnetic circuit enclosing a square solid with a hollow part shown in Fig. 2, then this square solid can be divided into 4 same trapezoidal solids shown in Fig. 3. Further, each of the trapezoids is divided into the 2 slices along with a diagonal surface as shown in Fig. 4. After that, the magnetic resistance to each of the divided trapezoids is further divided into mth subdivisions in a horizontal direction. Thus, an entire magnetic resistance of subdivided triangular solid is computed by connecting mth magnetic resistances in parallel shown in Fig. 5. Equation (8) calculates the composed magnetic resistance in Fig. 5.

$$\because \frac{1}{R_{\mathrm{TOTAL}}} = \sum_i \frac{1}{R_i} = \sum_{i=0}^{m} \frac{\mu B \cdot C/m}{\frac{A-2C}{2} + (\frac{A}{2} - \frac{A-2C}{2}) \cdot (i-1)/m}. \tag{8}$$

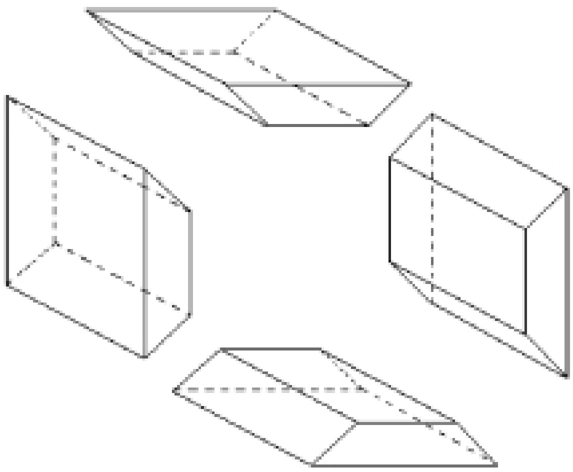

Fig. 3. Subdivision for four trapezoids.

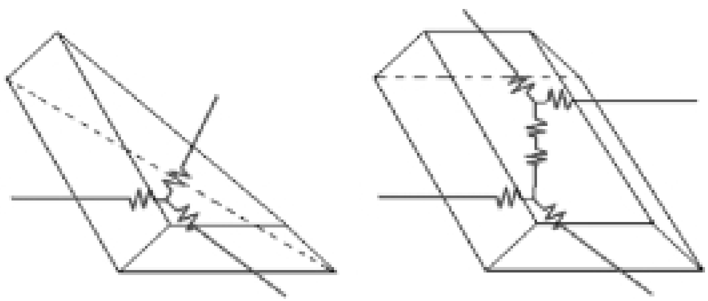

Fig. 4. Subdivision of trapezoid and magnetic circuit connection.

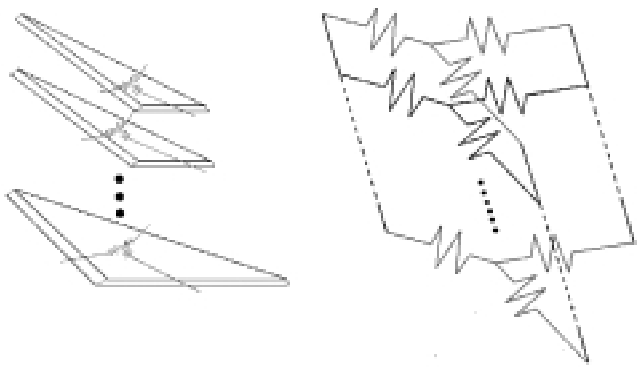

Fig. 5. The magnetic resistance of whole triangular solid.

In Eq. (8), let be $m \to \infty$, $C/m = dx$, $i/m = x$ then it is possible to apply the Riemann integral. This leads to

$$R_{\mathrm{TOTAL}} = \frac{C}{\mu B \log\left(\frac{A-2C+2C^2}{A-2C}\right)}. \tag{9}$$

Equation (9) is the magnetic resistance to the half of an original trapezoid. Since the original trapezoid is composed of the two same shapes of subdivisions, then whole trapezoidal magnetic resistance is given by Eq. (10).

$$R_{\mathrm{Whole-trapezoid}} = \frac{2C}{\mu B \log\left(\frac{A-2C+2C^2}{A-2C}\right)}. \tag{10}$$

Similarly, the magnetic resistances of the other trapezoids can be calculated, and finally whole enclosing magnetic resistance of the hollow square cubic shown in Fig. 2 is given by Eq. (11), because 4 magnetic resistances in Eq. (10) are connected in series.

$$R_{\mathrm{Whole}} = \frac{8C}{\mu B \log\left(\frac{A-2C+2C^2}{A-2C}\right)}. \tag{11}$$

Here, we generalize from a hollow quadrangle to the hollow n-sided polygons. First, it is carried out that a hollow n-sided polygon is divided for n^{th} trapezoids, because a hollow quadrangle is divided into 4 same trapezoids. In much the same way as the hollow quadrangle, the n^{th} trapezoids is divided into the 2 slices along with a diagonal surface. Thus, a general formula that evaluates the magnetic resistance of the half of an original trapezoid is given by Eq. (12).

$$R_{n-5} = \frac{1}{\int_0^C \frac{\mu B}{\left(\frac{2A}{n} - \frac{C}{\tan\left[\frac{(n-2)\pi}{2n}\right]}\right) + \left[\frac{2A}{n} - \left\{\frac{2A}{n} - \frac{C}{\tan\left[\frac{(n-2)\pi}{2n}\right]}\right\}\right]x} dx} \tag{12}$$

By means of Eq. (12), the whole magnetic resistance of a hollow n-sided polygon is given by Eq. (13).

$$R_{n-\mathrm{total}} = R_{n-5} \times 2 \times n. \tag{13}$$

When we consider about the physical dimension of the magnetic resistance from Eq. (13), it is obvious to be an inverse of inductance, i.e., 1/H or A/Wb. In other words, we can confirm that the magnetic resistance is the reciprocal number of inductance. Thus, after evaluating the magnetic resistance by our method, let us compare this with those of analytical one. In a hollow quadrangle shown in Fig. 2 assuming the constants: $A = 1$ m, $B = 1$ m, $C = 0.49$ m and permeability $\mu = 1$ H/m. The number of polygons is 512. A set of the parameters: $A = 1$ m, $B = 1$ m, $C = 0.49$ m and 512 polygon yields the maximum inductance 32.3 mH. Equation (14) gives the analytical value of inductance having solid cylindrical shape.

$$L = \frac{1}{2\pi}\left\{\frac{\mu}{4} + \mu_0\left(\log\frac{2l}{a} - 1\right)\right\} = 39.8\ \mathrm{mH} \tag{14}$$

Comparison the analytical inductance 39.8 mH with computed one 32.3 mH reveals that the first one figure is the same even if a hollow 512 polygon is used for computation. In order to take a flux distribution into consideration, we assumed the multi-layered flux paths. Figure 6 shows a relationship between the number of layers and inductance. From Fig. 6, we confirm that the inductance converges to the analytical solution by increasing the number of layers.

Thus, validity of the quasi-three-dimensional magnetic resistance by our TMMC has been verified.

Computation of the magnetic fluxes flowing through the multi-layer magnetic resistances gives a flux distribution. Figure 7 shows the magnetic flux distributions when changing the number of layers. From the results in Fig. 7, it is obvious that a large number of layers give a fine magnetic flux distribution.

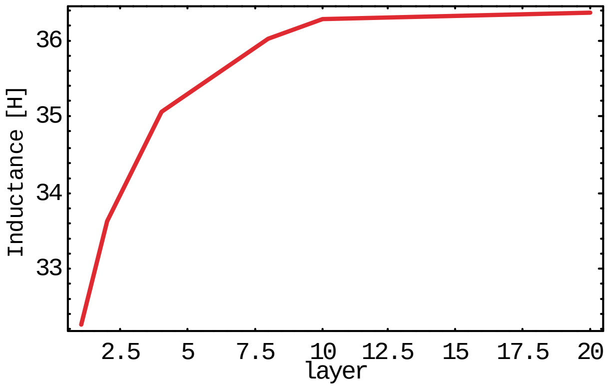

Fig. 6. Convergence process of inductance.

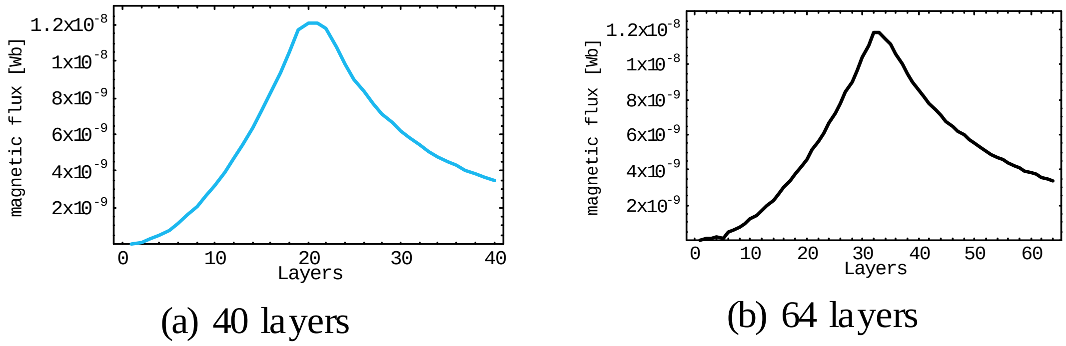

Fig. 7. Magnetic flux distributions.

3. Conclusion

In this paper, we have proposed the modern magnetic circuit theory based on the finite elements. At first, we have described to the two-dimensional theory of modern magnetic circuits. Second, we have generalized our theory of modern magnetic circuits into quasi-three-dimensional one. Generalization from two- to quasi-three-dimensional magnetic resistance has been reduced into Riemann integral.

To verify our methodology, we have calculated the inductance along with a simple example. It has been found that plural subdivided magnetic circuits connected in parallel give good inductance value as well as fine magnetic flux distribution compared with these of analytical ones.

References

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